Two School Systems, One District: What to do when a unified admissions process is impossible*

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Abstract

When groups of schools within a single district run their admission processes independently of one another, the resulting match is often inefficient: many children are left unmatched and seats are left unfilled.

We study the problem of re-matching students to take advantage of these empty seats in a context where there are priorities to respect. We propose an iterative way in which each group may independently match and re-match students to its schools.

The advantages of this process are that every iteration leads to a Pareto improvement and a reduction in waste while maintaining respect of the priorities. Furthermore, it reaches a non-wasteful match within a finite number of iterations.

While iterating may be costly, as it involves asking for inputs from the children, there are significant gains from the first few iterations. We show this analytically for certain stylized problems and computationally for a few others.

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1 Introduction

Across the United States there are many instances where schools are broken into groups which are antagonistic towards one another. The fault lines are often around charter and voucher systems.

The earliest voucher systems date back to the 19^{th} century when Maine and Vermont funded students from sparsely populated areas, far from public schools, to be educated at private schools. Currently twelve states and Washington D.C. offer voucher programs. Typically, a voucher covers a low income child's expenses should he or she attend a private rather than a public school.

Charter schools are publicly funded but independently administered schools. Since the 1970s, they have become increasingly common. In New Orleans, LA the majority of children are educated at charter schools (Mullins 2014). All but eight states currently have laws permitting charter schools.

While the specifics regarding eligibility and implementation of these programs vary between school districts, there is one important feature that is typical: The processes of matching students to different groups of schools are independent of one another. The parents of children who are matched to schools by both processes make a choice at the very end. It is not hard to see that this leads to inefficiencies: seats in one school system are "wasted" on children who end up choosing to go to the other system. Consequently, at the end, many children are left unmatched or matched to schools that they find worse than others with empty seats. In Milwaukee, WI, which has a voucher system, many students are left to participate in what is called "open enrollment" at this stage. This is an uncoordinated and chaotic process where parents apply to individual schools in search of a seat for their child. This haphazard way of matching children to schools is inefficient and leads to violations of priorities that children have at schools. In New Orleans many of these children simply do not show up on the first day of classes (Dreilinger 2013b).

From the sizable literature on school choice, starting with Abdulkadiroğlu and Sönmez (2003), it is clear that the "right" solution for this problem is a unified process of matching children to schools using the "child-optimal fair rule."[1]

Such a unified process, however, is not always possible. In Milwaukee, each public school is funded proportionally to the number of students it enrolls.The public school system (Milwaukee Public Schools or MPS) loses money for every student that is enrolled at a private school. Similarly, private schools (Milwaukee Parental Choice Program or MPCP) lose money for every student that is enrolled at a public school. The amount at stake per student was $6,442 for the 2013-14 school year (Wisconsin Statute 119.23).

Below we quote a Milwaukee based journalist who focuses on the local education system.[2]

> "I do not foresee MPS working with MPCP because they are both competing for the same crop of school-aged, Milwaukee resident children. And each has a vested interest in getting as many kids as possible to attend their respective schools because each kid gives them state money."
>
> — *Erin Richards of the Milwaukee Journal Sentinel.*[3]

This adversarial relationship between MPS and MPCP makes it politically infeasible to impose a unified matching process. It is possible, however, to control the separate processes that the two groups use to match children to their schools. Currently, both are required to match children in a way that takes certain priorities into account.[4] Neither group uses a well designed matching process. Private school admissions are entirely decentralized. Public schools use

[1] This rule, often called the "Deferred Acceptance" rule, has repeatedly turned out to be attractive from both efficiency and strategic points of view when there are fairness constraints in terms of school priorities over children (Abdulkadiroğlu, Pathak, Roth and Sönmez 2005, Abdulkadirolu, Pathak and Roth 2005, Kojima and Manea 2010, Ehlers and Klaus 2013, Abdulkadiroğlu, Pathak and Roth 2009, Kesten and Kurino 2013, Alva and Manjunath 2015).

[2] Neither MPS nor MPCP have responded to requests for comments.

[3] This quote is from an email correspondence.

[4] These are orderings over the children that dictate precedence of the children at various schools. They are usually based on coarse criteria with random tie-breaking. Even private schools are required to respect lottery based priorities when there are more applicants than available seats.

a variant of the "Boston rule" (Abdulkadiroğlu and Sönmez 2003) where each child only ranks three alternatives.

In New Orleans, though most schools are chartered they are administered by two different bodies: the Orleans Parish School Board and the Recovery School Board. Attempts to unify the admissions processes have failed (Dreilinger 2013b).[5] The issue of wasted seats has turned out to be severe, resulting in enrolled children failing to show up on the first day of classes (Dreilinger 2013a).

While a unified admissions process is typically not possible, that the two groups can be required to follow certain rules (such as using priorities to break ties) suggests that it may be reasonable to expect that, in many cases, the authorities could require each group of schools to use the child-optimal fair rule restricted to that group.

This is still not enough. Once the various[6] groups of schools separately match students using the child-optimal fair rule, the problem of wasted seats persists. In fact, as long as there are priorities that are to be respected, we show that it persists no matter what separate matching rules the groups of schools use (Theorem 1).

We are interested in finding a good way to deal with these vacant seats. The authorities may be able to dictate how the children are "re-matched" to schools to take advantage of vacant seats after the first match. We offer an iterated process where the groups of schools separately match and re-match students to schools until there are no wasted seats.[7] We are unaware of other work on the topic of solving parts of a single matching problem in parallel.[8]

The process that we prescribe can be thought of as being "slightly decentralized." Admission to each group of schools is done in a centralized way specific

[5]Ekmekci and Yenmez (2014) offer an explanation of why such attempts have failed in New Orleans and other cities. They also explore ways to provide incentives for charter schools to join a centralized clearinghouse.

[6]The analysis in the paper applies even to the case of more than two groups of schools.

[7]Dur and Kesten (2013) study a model of "sequential assignment" where two assignment problems are solved one after the other. The solution of the second problem takes as an input the allocation chosen for the first one. Their concern is with the strategic implications of matching in two steps rather than one. They provide conditions on sequential solutions to guarantee that the subgame perfect Nash equilibrium outcomes of the preference revelation game are fair.

[8]Anno and Kurino (2014) study the parallel solution of separate problems involving the same people.

to that group.[9] These various groups, however, are granted a measure of autonomy as they may run admissions algorithms for their schools independently.

Our process is as follows. Each child submits to each group of schools his preferences over those schools. Each group of schools separately applies the child-optimal fair rule restricted to that group of schools. A child who is matched to a school in more than one group accepts his most preferred among them and turns down the rest. This is the initial match. There may be wasted seats at this point since some of the matches are turned down. Each group of schools computes a re-match in the following way.

1. If a child accepted the seat offered by this group, his preferences over this group are "truncated" *at* the school which was offered to him. That is, every other school that he finds worse than what he has been offered is considered unacceptable. If a child turned down the seat offered by this group, his preferences over this group are truncated *above* the school that was offered to him. That is, the school that was offered to him and every school worse than it is deemed unacceptable.

2. The group "re-matches" using the child-optimal fair rule restricted to that group for the truncated preferences.

Again, every child who is matched to a school by more than one group accepts his most preferred among them and turns down the rest. This is the "re-match."

The re-match may still have wasted seats so the re-matching process (truncation of preferences and application of the matching rule to the truncated preferences) may be repeated. We show that if it is repeated enough times, no seats are wasted (Theorem 3).

The benefits of this process are:

1. Every match along the way respects priorities (that is, it is "fair") (Proposition 1).

[9] A "decentralized" method might involve "wait lists." Such a process would be fraught with the kinds of coordination problems described above. Our goal is to depart from a fully centralized process in a way that the groups of schools have a sense of autonomy yet achieve a non-wasteful match.

2. Every successive re-match is a Pareto-improvement over the previous match. That is, no child is ever made worse off after a re-match (Proposition 2).

Iterating this process long enough eliminates waste. It yields a fair, individually rational, and non-wasteful match that may be distinct from the one chosen by the unified child-optimal fair rule (Example 1).[10] [11] However, it may be reasonable to stop short of iterating until all wasted seats are eliminated since the resulting match will be fair for those who are matched. Asking every child to pick one among several schools offered to him may be a costly exercise. We show that re-matching even once or twice can go a long way. What we are interested in is "how close" a match is to being non-wasteful. In order to quantify this, we consider situations where there are precisely as many seats as their are children. For such problems, every child is assigned to some school at every fair and non-wasteful match. So the number of children that a match leaves unassigned is a way of quantifying how far a match is from being non-wasteful.[12] When there are only two groups of schools, for the case of identical preferences and identically and independently drawn priorities, we show that re-matching once can reduce the proportion of wasted seats from 25% to 9.4%. A second re-match can bring this down to about 4% and a third to below 2% (Theorem 4). We also consider the case where preferences are diverse in the sense of every preference relation being represented in the same proportion. If priorities are still drawn independently and identically, then the proportion of wasted seats can be decreased from 25% to 11%, 6.25%, and 4% after one, two, and three rematches respectively (Theorem 5). Since the analytical approach is intractable for more general preferences, we use computer simulations to compute the proportion of wasted seats when preferences are described by a convex combination of common and independently distributed private values. These confirm the result that there are significant benefits from the first few iterations (Table

[10]The "unified child-optimal fair rule" is where the admissions problem is unified and the deferred acceptance algorithm is run with the priorities of all of the schools and preferences of children over all of the schools as inputs.

[11]We are not aware of other polynomial time algorithms to compute fair, individually rational, and non-wasteful matches that yield neither the child-optimal nor child-pessimal fair match.

[12]For our ordinal setting, this is a reasonable "social welfare function."

2).

Finally, is it in the schools' best interest to re-match in this way? Our results imply that if a fair match is wasteful, then at least one group of schools has an incentive to re-match. That is, at least one group of schools would strictly increase the number of students assigned to it by re-matching.

The remainder of the paper is organized as follows. In Section 2 we formally define our model. In Section 3 we consider the implications of the groups of schools matching separately. In Section 4 we discuss the unified child-optimal fair rule. In Section 5 we formally describe our process of separate matching and re-matching. In Section 5.1 we show that this process has desirable fairness and efficiency properties. In Section 6 we show that there are significant benefits even from a few iterations of the re-matching process. We do this analytically in Section 6.1 and computationally in Section 6.2. Some concluding remarks are in Section 7. All proofs are in the Appendix.

2 The model

The set of schools, S, is partitioned into K subsets $\{S^k\}_{k\in K}$. For each $s \in S$, q_s is the capacity of school s. The set of children is C. We use $\varnothing$ to denote "not going to school." For each $c \in C$, P_c is c's preference relation over $S \cup \{\varnothing\}$. We assume that P_c is a linear order. The set of all such preference relations is $\mathcal{P}$.

For each $s \in S$, $\succ_s$ is a linear order over C. It is the "priority" order associated with s. For each $k \in K$, $\mathcal{P}^k$ is the set of preference relations over $S^k \cup \{\varnothing\}$. For $P_c \in \mathcal{P}$, we denote its restriction to $S^k \cup \{\varnothing\}$ by $P_c^k \in \mathcal{P}^k$. In other words, P_c^k is such that for each pair $s, s' \in S^k \cup \{\varnothing\}, s \; P_c^k \; s'$ if and only if $s \; P_c \; s'$. We use $s \; R_c \; s'$ to mean that either $s \; P_c \; s'$ or $s = s'$.

For each $s \in S$, $U(P_c, s)$ is the weak upper contour set of P_c at s and $\overline{U}(P_c, s)$ is the strict upper contour set of P_c and s. That is,

$$\begin{aligned} U(P_c, s) &= \{s' \in S : s' \; R_c \; s\}, \text{ and} \\ \overline{U}(P_c, s) &= \{s' \in S : s' \; P_c \; s\}. \end{aligned}$$

For each $k \in K$ and each $s \in S^k \cup \{\varnothing\}$, we define $U(P_c^k, s)$ and $\overline{U}(P_c^k, s)$

similarly.

A match is $\mu : C \to S \cup \{\varnothing\}$ such that for each $s \in S$, $|\mu^{-1}(s)| \leq q_s$. Since there will be no confusion, from now on we will omit the superscript "-1" and use $\mu(s)$ to mean $\mu^{-1}(s)$. The set of all matches is $\mathcal{M}$.

For each $k \in K$, $\mathcal{M}^k$ is the set of matches where each child is either matched to a school in S^k or to $\varnothing$. That is,

$$\mathcal{M}^k = \{\mu \in \mathcal{M} : \text{ for each } c \in C, \mu(c) \in S^k \cup \{\varnothing\}\}.$$

Before proceeding, we describe some properties of a match. The first says that a seat is "wasted" if it is left empty despite there being a child who prefers it to what he is matched to. That is, for each $P \in \mathcal{P}^C$ and each $\mu \in \mathcal{M}$, μ is *wasteful at* P if there are $c \in C$ and $s \in S$ such that $s \; P_c \; \mu(c)$ and $|\mu(s)| < q_s$.

The second says that no child is assigned to a school that he finds worse than not going to school at all. That is, for each $P \in \mathcal{P}^C$ and each $\mu \in \mathcal{M}$, μ is *individually rational at* P if for each $c \in C, \mu(c) \; R_c \; \varnothing$.

The last property says that the priorities are "respected." That is, for each $P \in \mathcal{P}^C$ and each $\mu \in \mathcal{M}$, μ *is fair at* P (Balinski and Sönmez 1999) if for each $c \in C$ and $s \in S$ such that $s \; P_c \; \mu(c)$, for each $c' \in \mu(s), c' \succ_s c$.[13]

For $P \in \mathcal{P}^C$, we denote by $NW(P), I(P)$, and $F(P)$ the sets of non-wasteful matches, individually rational matches, and fair matches at P.

The following concept is useful in comparing the welfare of children under two different matches. It says that one match is better than another match if every child finds his assignment at least as desirable and at least one child finds his assignment more desirable under the first match than the second. That is, for $P \in \mathcal{P}^C$ and each pair $\mu, \mu' \in \mathcal{M}$, μ *Pareto-dominates* μ' *at* P if for each $c \in C, \mu(c) \; R_c \; \mu'(c)$ and for some $c \in C, \mu(c) \; P_c \; \mu'(c)$.

[13]Some authors define fairness as "stability" which is the combination of what we have called fairness, individual rationality, and non-wastefulness.

3 Separate matching

For each $k \in K$, the procedure the k^{th} group of schools, S^k, uses to match children to schools in S^k takes as an input each child's preferences over S^k and returns a match in $\mathcal{M}^k$. This defines an S^k*-rule*, $\varphi^k : \mathcal{P}^{kC} \to \mathcal{M}^k$.

When each group of schools separately applies its own rule, each child may be matched to more than one school. In practice, this is resolved by letting each child pick his most preferred school among those he is matched to. We define a way of combining matches in this way. For a set of matches $M \subset \mathcal{M}$ and a profile of preferences $P \in \mathcal{P}^C$, the "sup" of M at P, which we denote by $\mathcal{C}(M, P)$, is $f : C \to S \cup \{\varnothing\}$ such that for each $c \in C$, $f(c)$ is c's most preferred school among those that he is matched to by matches in M. That is,

$$f(c) = P_c\text{-max}\{\mu(c) : \mu \in M\}.^{14}$$

It is obvious that if $M = \{\mu^k\}_{k \in K}$ where, for each $k \in K$, $\mu^k \in \mathcal{M}^k$ then $\mathcal{C}(M, P) \in \mathcal{M}$.

It is useful to observe that individual rationality and fairness of matches are preserved by $\mathcal{C}$. These follow from the well known result attributed to John Conway by Knuth (1976).

Observation 1. *The sup of individually rational S^k-matches is individually rational.*

Observation 2. *The sup of fair S^k-matches is fair.*

When, for each $k \in K$, S^k uses φ^k and preferences are $P \in \mathcal{P}^C$, the set of matches to be combined is

$$\{\varphi^1(P^1), \varphi^2(P^2), \ldots, \varphi^K(P^K)\}.$$

Thus, after each group has separately applied its own rule, the resulting match is $\mathcal{C}(\{\varphi^1(P^1), \ldots, \varphi^K(P^K)\}, P)$.

In general, any process which takes as input children's preferences over all schools and outputs a match in $\mathcal{M}$ defines a *rule*, $\varphi : \mathcal{P}^C \to \mathcal{M}$. If φ is the sup of S^k-rules as defined above, we say that it is *separable*.

[14] For $P_c \in \mathcal{P}$ and $S' \subseteq S$, P_c-max $S' = s \in S'$ such that for each $s' \in S', s \; R_c \; s'$.

A rule is individually rational if it picks an individually rational match for every problem. A rule is fair if it picks a fair match for every problem.

If no cooperation is possible across the groups of schools, whoever picks the rule to apply faces a constraint that the rule be separable. If this decision maker is also constrained by fairness, there is no way around being wasteful.

Theorem 1. *If a rule is fair and separable then it does not pick a non-wasteful match for every problem.*

4 The unified child-optimal fair rule

For each problem, there is a unique fair, individually rational, and non-wasteful match that Pareto-dominates every other fair, individually rational, and non-wasteful match (Gale and Shapley 1962). In fact, for each $P \in \mathcal{P}^C$, there is $\mu^{CO} \in F(P)$ such that for every $\mu \in F(P)$, μ^{CO} Pareto-dominates μ at P. It is easy to see that $\mu^{CO} \in I(P) \cap NW(P)$. We call it the "child-optimal fair match at P." That $F(P) \cap I(P) \cap NW(P)$ is a lattice, and therefore has a Pareto-best and a Pareto-worst element, is well known. Theorem 2 is a slight generalization to show that the Pareto-best element of $F(P) \cap I(P) \cap NW(P)$ Pareto-dominates every element of $F(P)$. Fairness, as we have defined it, is weaker than "stability" (which is fairness along with non-wastefulness and individual rationality).

The rule that, for each $P \in \mathcal{P}$, selects the child-optimal fair match at P is the "child-optimal fair rule," φ^{CO}. Starting with Abdulkadiroğlu and Sönmez (2003), the literature on school choice has largely endorsed this rule.

As we have argued, when the matching problem cannot be unified, φ^{CO} is not an option. However, we use it as a benchmark.

Theorem 2. *The child optimal fair match Pareto-dominates every other fair match.*

The following, a corollary to Theorem 1, says that φ^{CO} is not separable.

Corollary 1. *The child-optimal fair rule is not separable.*

This tells us that we cannot find a φ^k for each $k \in K$ such that when each S^k applies the rule φ^k separately, the sup would yield φ^{CO}. That is, the ideal solution cannot be attained in a separated regime.

5 Separate matching and re-matching

From Corollary 1, we see that φ^{CO} is not separable. In this section we define an iterated procedure for matching and re-matching.

Each group of schools uses the child-optimal fair rule restricted to its schools. The resulting matches are combined by letting each child pick among the schools that he is matched to. Then, children are matched for a second time by each group of schools. These are, again, combined and the process continues. While this iterated process does not define a separable rule, the matching and re-matching processes are done separately. Following is a precise definition of the iterated process which defines the *L-iterated child-optimal rule, $\varphi^{L\text{-}CO}$*. That is, we define the result of an initial match followed by $L-1$ re-matches.

We start with some notation. Suppose that $P_0 \in \mathcal{P}$. For each $S' \subseteq S$, we define the *truncation of P_0 with respect to S'* as $P_0|_{S'} \in \mathcal{P}$ such that:

1. It agrees with P_0 over $S' \cup \{\varnothing\}$. That is, for each pair $s, s' \in S' \cup \{\varnothing\}$, $s \; P_0|_{S'} \; s'$ if and only if $s \; P_0 \; s'$.

2. It ranks every $s \in S \setminus S'$ below $\varnothing$. That is, for each $s \in S \setminus S'$, $\varnothing \; P_0|_{S'} \; s$.

For each $k \in K$, let ψ^{kCO} be the child-optimal fair S^k-rule.[15] We define $\varphi^{L\text{-}CO}$ by setting, for each $P \in \mathcal{P}^C$, $\varphi^{L\text{-}CO}(P) = \mu_L$ where μ_L is computed as follows.

$$
\begin{array}{ll}
\text{For each } c \in C, & \mu_0(c) = \varnothing, \\
\text{for each } l \in \{1, \ldots, L\}, & S_c^l = U(P_c, \mu_{l-1}(c)), \\
 & P_c^l = P_c|_{S_c^l}, \\
\text{for each } k \in K, & \mu_l^k = \psi^{kCO}((P^l)^k), \text{ and} \\
 & \mu_l = \mathcal{C}(\{\mu_l^k\}_{k \in K}, P).
\end{array}
$$

An interpretation of this computation is the following: Each $c \in C$, for each $k \in K$, submits his preferences P_c^k to S^k. Then, S^k applies ψ^{kCO} to P^k, resulting in μ_1^k. It offers $\mu_1^k(c)$ to c. Having received, for each $k \in K$, an offer $\mu_1^k(c)$ (which may be $\varnothing$) from S^k, c chooses his most preferred among them. Each S^k now

[15] This selects the child optimal fair match for the problem $(P^k, \succ_{S^k})$.

re-matches by truncating, for each $c \in C$, c's submitted preference P_c^k: If he accepted $\mu_1^k(c)$, then anything worse than it is deemed unacceptable according to the truncated preference. If he turned it down, then $\mu_1^k(c)$ and anything below it is deemed unacceptable according to the truncated preference. With these truncated preferences, $(P_c^2)^k$, for each $c \in C$, S^k applies ψ^{kCO} to re-match children. This new match is μ_2^k and S^k offers $\mu_2^k(c)$ to c. Again, each child chooses from the offers that he receives. The process is repeated $L - 2$ more times.

5.1 Fairness, individual rationality, and waste

For fixed L, the match that $\varphi^{L\text{-}CO}$ picks, though fair and individually rational (Proposition 1), may be wasteful. However, an important feature of $\varphi^{L\text{-}CO}$ is that increasing L leads to Pareto-improvements (Proposition 2). In fact, fewer children are unmatched as L increases. (Proposition 3). The last result of this section is that for each $P \in \mathcal{P}^C$, if L is large enough, $\varphi^{L\text{-}CO}(P)$ is not wasteful at P (Theorem 3).

First, we show that the L-iterated child optimal rule picks a fair and individually rational match for every L.

Proposition 1. *The L-iterated child optimal rule is fair and individually rational.*

We next show that it is Pareto-improving to increasing the number of iterations.

Proposition 2. *For every L and every profile of preferences, the $(L+1)$-iterated child optimal rule picks either the same match as or a match that Pareto-dominates the one chosen by the L-iterated child optimal rule.*

Next, we consider "waste." This is tied to the number of matched students after every iteration.

Observation 3. *If μ and μ' are individually rational and μ Pareto-dominates μ', then any child unmatched by μ is unmatched by μ'.*

Observation 3 and Theorem 2 together suggest a natural benchmark for the number of students who are unmatched at a given $P \in \mathcal{P}^C$. Since $\varphi^{CO}(P)$ is not wasteful, individually rational, and fair, any child left unmatched is due to the fairness constraints. Thus, if we insist on fairness, this number of unmatched children is unavoidable. So we use the number of students left unmatched at μ^{CO} as a benchmark and define the following measure of wastefulness. For each $P \in \mathcal{P}^C$ and each $\mu \in M$,

$$W(\mu, P) = |\{c \in C : \mu(c) = \varnothing\}| - |\{c \in C : \varphi^{CO}(P)(c) = \varnothing\}|.$$

Our next result says that waste decreases as number of iterations increases. It follows immediately from Observation 3 and Proposition 2.

Proposition 3. *For every L and every profile of preferences, the $(L+1)$-iterated child optimal rule picks a match that is no more wasteful than the match chosen by the L-iterated child optimal rule.*

If the number of iterations is high enough, waste can be eliminated altogether.

Theorem 3. *For every profile of preferences, if the L is large enough, the L-iterated child optimal rule picks a non-wasteful match.*

It is worth pointing out that even though $\varphi^{L\text{-}CO}(P)$ is fair, individually rational, and non-wasteful at P when L is large enough, it need not coincide with $\varphi^{CO}(P)$. Example 1 demonstrates this.

Example 1. *Even for large L, the L-iterated child optimal rule may not coincide with child-optimal fair rule.*

Consider a problem where $S = \{s_1, s_2\}$, $C = \{c_1, c_2\}$, and $K = 2$ so that $S^1 = \{s_1\}$ and $S^2 = \{s_2\}$. For each $s \in S$, $q_s = 1$. Suppose that $c_1 \succ_{s_1} c_2$ and $c_2 \succ_{s_2} c_1$. Let $P \in \mathcal{P}^C$ be such that $s_2 \ P_{c_1} \ s_1 \ P_{c_1} \ \varnothing$ and $s_1 \ P_{c_2} \ s_2 \ P_{c_2} \ \varnothing$. Then $\varphi^{CO}(P)(c_1) = s_2$ and $\varphi^{CO}(P)(c_2) = s_1$.

We first show that, in the definition of $\varphi^{L\text{-}CO}$, μ_1 is such that $\mu_1(c_1) = s_1$ and $\mu_1(c_2) = s_2$. For each $k \in K$, and each $c \in C$, P^k_c is such that $s_k \ P^k_c \ \varnothing$. By definition of ψ^{1CO}, $\mu^1_1(c_1) = s_1$ and $\mu^1_1(c_2) = \varnothing$. Similarly, $\mu^2_1(c_1) = \varnothing$ and $\mu^2_1(c_2) = s_2$. Since, $\mu_1 = \mathcal{C}(\{\mu^1_1, \mu^2_1\}, P)$, $\mu_1(c_1) = s_1$ and $\mu_1(c_2) = s_2$.

Since truncation does not change the preferences, $\mu_2 = \mu_1$. Thus, for every L, $\varphi^{L\text{-}CO}(P)(c_1) = s_1$ and $\varphi^{L\text{-}CO}(P)(c_2) = s_2$ so that $\varphi^{CO}(P) \neq \varphi^{LCO}(P)$. ◦

In Example 1, $\varphi^{L\text{-}CO}$ selects the "school-optimal fair match." In general, if for each $k \in K$, S^k is a singleton, (that is, $\{S^k\}_{k \in K}$ is the finest partition of S), then the iterative definition of $\varphi^{L\text{-}CO}$, if L is large enough, is equivalent to the "school proposing deferred acceptance" algorithm.

Remark 1. Matching and re-matching as many times as needed to arrive at a non-wasteful match yields a new rule that has not appeared in the literature before. It is a fair, non-wasteful, and individually rational rule which is distinct from both the child or the school optimal-rules. Yet, computing it does not require enumeration of the entire "stable set" as is the case with the "median stable rule" (Teo and Sethuraman 1998) which is one of the few other such rules discussed in the literature. Since it picks neither child nor school optimal fair match, it favors neither side as much. In applications where both sides have preferences (rather than priorities on one side), this could represent a meaningful compromise. See Kuvalekar (2015) for another instance of such rules.

6 Is it worth iterating just a few times?

In this section we consider how the match evolves with successive iterations. We take two approaches. The first is analytical and the second is computational. With either approach, we look at situations where there are only two groups of schools. Our main finding is that there are significant gains from the first few iterations.

Analytically, we calculate the progression of matches for two specific problems where priorities are identically and independently distributed. The first is where each child has the same preferences and the second is where preferences are as diverse as possible.

Since we are interested in determining *how close* the match gets to being non-wasteful, we need a way of quantifying this. To do that, we consider situations where there are as many school seats as there are children. In any fair and non-wasteful match, every child is assigned a seat and every seat is filled. The number of children that a match leaves unassigned is thus a measure of its distance from non-wastefulness.

Our answer to the question of how many children are left unmatched after each round depends on the number of schools when preferences are identical. When they are diverse, it does not. We solve for these numbers in a model where the set of children is a continuum and a school's capacity is the mass of students that it can admit.

The limitation of the analytical approach is that it is not tractable for a wider class of preference profiles. This is where we turn to the computational approach. We again study economies where priorities are identically and independently distributed. Preferences are described by a weighted average of a common value and an identically and independently distributed idiosyncratic value and thus convex combinations of the cases that we study analytically.[16]

Both approaches suggest that the first few iterations exhaust much of the benefits from iterating. Thus, $\varphi^{\text{2-CO}}$ improves greatly over $\varphi^{\text{1-CO}}$. Even if it is too costly to re-match more than once, just doing it once has significant benefits.

Remark 2. For the rest of this section, we consider the case where there are exactly as many seats as there are children and that each school is acceptable to each child and vice versa. In most school choice problems, there are enough seats in total to guarantee that every child is matched. If, nonetheless, seats were scarce, the problem of wasted seats would be worse than the case that we study.

6.1 The analytical approach

To facilitate our analytical exercise, we use a continuum framework (Abdulkadiroğlu et al. forthcoming, Azevedo and Leshno 2013).

Each child is described by a preference relation in $\mathcal{P}$ and a cardinal priority at each school. That is, each child is associated with an element of $C = \mathcal{P} \times [0,1]^S$. We will assume that the profile of preferences and priorities for an economy is then described by a nonatomic unit measure ρ over C.[17] Each school's capacity,

[16]See Abdulkadiroğlu, Che and Yasuda (forthcoming) and Hafalir, Yenmez and Yildirim (2013) for other simulations in the school choice context with similarly drawn preferences. See Lee (2013) for an analytical analysis under this assumption on the distribution of preferences.

[17]Consider a pair of children $c, c' \in C$ such that $c = (P_c, p_c)$ and $c' = (P_{c'}, p_{c'})$. For each

rather than being an integer as before, is $q_s \in [0,1]$. We only consider the case where there are two groups, S^1 and S^2, each consisting of m schools. That is, $|S^1| = |S^2| = m$. Further, we assume that each school has the same capacity and there are exactly as many seats as there are children. That is, for each $s \in S$, $q_s = \frac{1}{2m}$.

A match, $\mu : C \to S$, is a right continuous[18] and measurable function such that for each $s \in S$, $\rho(\mu(s)) \leq q_s$. As before, the set of matches is $\mathcal{M}$.

Any fair match is determined by a "cutoff" for each school (Balinski and Sönmez 1999, Adachi 2000, Azevedo and Leshno 2013). That is, for each fair $\mu \in \mathcal{M}$, there is $\kappa \in [0,1]^S$ where for each $c \in C$, $\mu(c) = P_c\text{-max}\{s \in S : p_{cs} \leq \kappa_s\}$.

A rule, $\varphi : \Delta(C) \to \mathcal{M}$, associates each problem with a match. The definitions of φ^{CO} and $\varphi^{L\text{-}CO}$ are readily adapted to this environment.

We study economies with independently and uniformly drawn priorities but look at two extreme cases for preferences. The first is where every child has the same preference relation and the other is where preferences are as diverse as possible.[19]

Since we have assumed that there are precisely as many seats at schools as there children, and that every school is acceptable to every child and vice versa, the number of unmatched children is particularly relevant. Under the assumption that priorities are uniformly drawn, given a profile of cutoffs $\kappa \in [0,1]^S$, it is easy to calculate the mass of children who are left unmatched. At κ, for each $s \in S$ and each $c \in C$, if $p_{cs} > \kappa_s$ then c is denied admission to s. The set of children who are denied admission to s at κ is then $\Delta(\kappa_s) = \{c \in C : p_{cs} > \kappa_s\}$. Since we assume that priorities are identically and uniformly distributed, $\rho(\Delta(\kappa_s)) = 1 - \kappa_s$. A child is unmatched if he is denied admission at every school. Thus, the set of children who are unmatched at κ is $\Upsilon_\kappa = \cap_{s \in S} \Delta(\kappa_s)$ and

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$s \in S$, we will say that c has a higher priority than c' at s if and only if $p_{cs} < p_{c's}$. Since we assume that ρ is nonatomic, only a zero-measure set of children are tied in s's priority order.

[18] Azevedo and Leshno (2013) define *right continuity* as follows. For each sequence $\{c^n\}_{n=1}^{\infty}$ in C such that $p_{c^n} \to p_c$ and for each $s \in S$ and n, $p_{c^n s} \leq p_{c^{n+1} s} \leq p_{cs}$, there is N such that $\mu(c^n) = \mu(c)$ for each $n \geq N$.

[19] Since we are working with a continuum of children this means that each preference relation is held by the same mass of children.

$\rho(\Upsilon_\kappa) = \Pi_{s\in S}(1-\kappa_s)$.

6.1.1 Identical preferences

There is $P_0 \in \mathcal{P}$ such that every child has P_0 as his preference relation. That is, $\rho(\{P_0\} \times [0,1]^S) = 1$.

The first thing to note is that there is a unique fair, individually rational, and non-wasteful match when every child has the same preference relation. Thus, a continuum version of Theorem 3 would say that as L grows, in the limit $\varphi^{L\text{-}CO}(\rho)$ converges to $\varphi^{CO}(\rho)$. In fact, this happens for finite L: it is bounded by the total number of schools. The exact L depends on P_0. For instance, if it ranks all schools in S^1 above all schools in S^2, then $\varphi^{2\text{-}CO}(\rho) = \varphi^{CO}(\rho)$.[20] If it ranks half the schools in S^1 above all the schools in S^2 but the other half of schools in S^1 below all the schools in S^2, then the smallest L such that $\varphi^{L\text{-}CO}(\rho) = \varphi^{CO}(\rho)$ is 3. Intuitively, the more "shuffled" the preference relation is, the larger this minimum L is. Thus, it is largest when P_0 is perfectly shuffled: it ranks the elements of S^1 and S^2 in an alternating way. That is, if $\{s_1, \ldots, s_{2m}\}$ is a labeling of S such that $s_1 \; P_0 \; s_2 \; P_0 \ldots P_0 \; s_{2m}$, then for each $k \in \{1, \ldots, m\}$, $s_{2k-1} \in S^1$ and $s_{2k} \in S^2$. As we show below, for such P_0, the smallest L such that $\varphi^{L\text{-}CO}(\rho) = \varphi^{CO}(\rho)$ is $2m$, the total number of schools.

Before solving for the cutoffs of $\varphi^{L\text{-}CO}(\rho)$, we solve for the cutoffs of $\varphi^{CO}(\rho)$ as this is our benchmark.

Proposition 4. *At $\varphi^{CO}(\rho)$, for each $k \in \{1, \ldots, 2m\}$, the cutoff for s_k is $\frac{1}{2m-k+1}$. Further, no child is left unmatched.*

Proposition 4 gives us benchmarks for the cutoffs. Since there is a unique fair, individually rational, and non-wasteful match at ρ, as L grows larger, the cutoffs of $\varphi^{L\text{-}CO}(\rho)$ converge to those of Proposition 4. Our goal here is to see, for each $L > 1$, how much closer to those in Proposition 4 the cutoffs of $\varphi^{L\text{-CO}}(\rho)$ are as compared to $\varphi^{1\text{-CO}}(\rho)$. To do this, we first calculate the cutoffs of $\varphi^{1\text{-CO}}(\rho)$.

[20] After the first round of $\varphi^{2\text{-}CO}$, the cutoffs for each school in S^1 coincide with their cutoffs under φ^{CO}. In the second round, no child assigned a seat at a school in S^1 in the first round finds a seat at any school in S^2 to be acceptable. Thus, the cutoffs for S^2 after the second round are what they would be under φ^{CO}.

Proposition 5. *At $\varphi^{1\text{-CO}}(\rho)$, for each $k \in \{1, \dots, m\}$, the cutoffs of s_{2k-1} and s_{2k} are both $\frac{1}{2m-k+1}$. Further, $\frac{1}{4}$ of the children are left unmatched.*

Next we calculate the cutoffs of $\varphi^{2\text{-CO}}(\rho)$.

Proposition 6. *At $\varphi^{2\text{-}CO}(\rho)$, the cutoff of s_1 is $\frac{1}{2m}$, the cutoff of s_2 is $\frac{1}{2m-1}$ and for each $k \in \{2, \dots, m\}$, the cutoffs of s_{2k-1} is*

$$\frac{1}{(2m-k+1)\left(1 - \sum_{j=2m-(k-2)}^{2m} \frac{1}{j}\right)}$$

and that of s_{2k} is

$$\frac{1}{(2m-k)\left(1 - \sum_{j=2m-(k-1)}^{2m-1} \frac{1}{j}\right)}.$$

A similar line of reasoning as in the proof of Proposition 6 allows us to calculate the cutoffs after L iterations. In this general case, closed form formulae as in Proposition 6 are not possible, so we only provide recursive formulae.

Theorem 4. *Let $L \geq 2$. At $\varphi^{L\text{-}CO}(\rho)$, for each $k \in \{1, \dots, L\}$,*

$$\kappa^L_{s_k} = \frac{1}{2m-k+1}$$

and for each $k \in \{L, \dots, 2m\}$,

$$\kappa^L_{s_k} \quad = \quad \frac{\kappa^L_{k-2}}{(1-\kappa^L_{s_{k-2}})(1-\kappa^{L-1}_{s_{k-1}})}.$$

In contrast with Proposition 5, it is not possible to pin down the exact proportion of unmatched children after $L \geq 2$ iterations independently of m. It can, however, be computed using the recursive formula for κ^L of Theorem 4. As shown in Table 1, it is increasing in m but remains lower than 0.095 even for very large values of m after the second iteration. After the third iteration, it drops to around 0.04 and below 0.02 after the fourth.

		L			
		1	2	3	4
m	10	0.25000	0.09315	0.04024	0.01668
	20	0.25000	0.09390	0.04088	0.01859
	40	0.25000	0.09410	0.04105	0.01913
	80	0.25000	0.09414	0.04109	0.01927
	160	0.25000	0.09415	0.04110	0.01931
	320	0.25000	0.09416	0.04110	0.01932
	640	0.25000	0.09416	0.04110	0.01932

Table 1: The proportion of unmatched children after L iterations when there are two groups of m schools and all children have the same "perfectly shuffled" preference relation over them.

6.1.2 Diverse preferences

We assume that ρ is such that for each pair $P_0, P_0' \in \mathcal{P}$, $\rho(\{P_0\} \times [0,1]^S) = \rho(\{P_0'\} \times [0,1]^S)$.[21]

As with the case of identical preferences, there is a unique fair, individually rational, and non-wasteful match at ρ (Azevedo and Leshno 2013). So a continuum version of Theorem 3 would say that as L grows larger, the cutoffs of $\varphi^{L\text{-}CO}(\rho)$ converge to those of $\varphi^{CO}(\rho)$. Unlike the case of identical preferences, however, they only converge in the limit. For no finite L does $\varphi^{L\text{-}CO}(\rho) = \varphi^{CO}(\rho)$. Nonetheless, we are interested in how quickly $\varphi^{L\text{-}CO}(\rho)$ converges to $\varphi^{CO}(\rho)$.

We start by calculating the cutoffs for $\varphi^{CO}(\rho)$.

Proposition 7. *At $\varphi^{CO}(\rho)$, for each $s \in S$, the cutoff for s is* 1. *Further, no child is left unmatched.*

Next, we calculate the cutoffs for $\varphi^{L\text{-}CO}(\rho)$. Unlike $\varphi^{CO}(\rho)$, for each L, $\varphi^{L\text{-}CO}(\rho)$ is wasteful.[22]

Theorem 5. *At $\varphi^{L\text{-}CO}(\rho)$, for each $s \in S$, the cutoff is $\left(1 - \frac{1}{(L+1)^{\frac{1}{m}}}\right)$. Further, $\frac{1}{(L+1)^2}$ of the children are left unmatched.*

[21] Since there are $2m$ schools, $|\mathcal{P}| = 2m!$. Thus, for each $P_0 \in \mathcal{P}$, $\rho(\{P_0\} \times [0,1]^S) = \frac{1}{2m!}$.

[22] This is because there is a continuum of children.

Unlike the case of identical preferences, when they are uniformly distributed, the cutoffs and the proportion of unmatched children does not depend on the number of schools.

6.2 The computational approach

In this section, we present the results of computer simulations. We analyze large (but finite) problems where priorities and preferences are drawn randomly.

We run our simulations with 160,000 children and 160 schools, each of which has 1000 seats, divided into two groups of 80. For each $s \in S$, we draw a *cardinal* priority over the students, p_s, such that for each $c \in C$, p_{cs} is independently and uniformly drawn between 0 and 1. Only the ordinal part of this is relevant: for each pair $c, c' \in C$, $c \succ_s c'$ if and only if $p_{cs} < p_{cs'}$.[23]

We draw *cardinal* preferences in the following way. For each $s \in S$, there is a common value component v_{0s} and for each $c \in C$, there is a private value component v_{cs}. For some fixed (and common) $\alpha \in [0, 1]$, the utility of c matched to s is

$$u_{cs} = \alpha v_{0s} + (1 - \alpha) v_{cs}.$$

For each $s \in S, v_{0s}$ and for each $c \in C, v_{cs}$ are also independently and uniformly drawn between 0 and 1. Again, only the ordinal aspect of u_c matters, so c's preference relation is $P_c \in \mathcal{P}$ such that for each pair $s, s' \in S$, $s \; P_c \; s'$ if and only if $u_{cs} > u_{cs'}$.[24]

The results of the simulations are summarized in Table 2 and visualized in Figure 1. We run our simulations on 1000 draws of the economy with each of five values for α: 0.0, 0.25, 0.5, 0.75, and 1.0. The table provides the average proportion of unmatched children for each draw of the economy.

[23] In the event of a tie, we break it randomly.
[24] In the event of a tie, we break it randomly.

		L			
		1	2	3	4
α	0.00	0.250043	0.119819	0.073324	0.050907
	0.25	0.250025	0.117597	0.070889	0.048607
	0.50	0.249995	0.107742	0.060116	0.038739
	0.75	0.249977	0.096342	0.045763	0.025247
	1.00	0.250014	0.093173	0.040138	0.018279

Table 2: The average proportion of unmatched children after L iterations for various values of α.

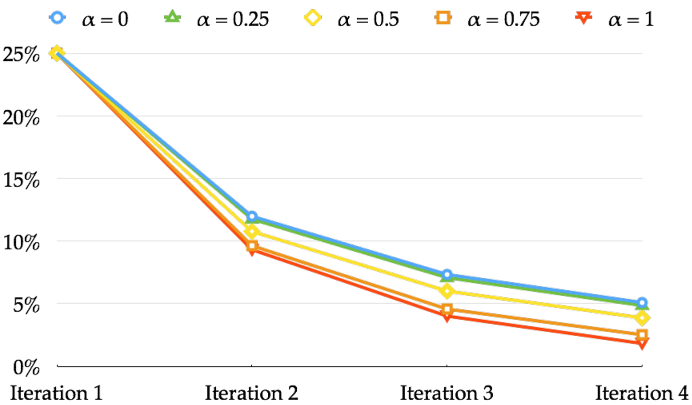

Figure 1: The vertical axis measures the percentage of children left unmatched. The horizontal axis measures the number of iterations. This is a plot of the data in Table 2

7 Conclusion

We have provided a way of dealing with groups of schools, who cannot have a unified matching process, by governing the way that they re-match. The main problem is that information about children's preferences across the groups of schools is unavailable. One might think of our process as a way of progressively revealing enough of this information bit by bit. As we have shown, the process that we have prescribed has desirable efficiency and fairness properties.

The main concern about applying such a process in the real world is that it is costly to re-match. So re-matching many times would be infeasible. Our results, both analytical and computational, suggest that even re-matching just a few times has great benefits. This goes a long way in assuaging concerns about the cost of re-matching.

An overriding concern in earlier work on school choice has been the requirement that the matching rules be "strategy-proof." That is, the requirement that children ought to not be able to gain by misreporting their preferences. Our process does not define a strategy-proof rule since it does not coincide with the child optimal fair rule. However, there is an objective sense in which it is difficult for a child to gain by manipulating: he would have to be able to see at least two steps ahead. Since the child optimal fair rule itself is strategy-proof, there is no opportunity for a child to misreport his preferences in a way that he receives a better match after the first match. It must be that he gains from further re-matching.

Appendices

A Proofs

A.1 Proof of Theorem 1

Proof of Theorem 1: Let $S = \{s_1, s_2\}$ and let $K = 2$ so that $S^1 = \{s_1\}$ and $S^2 = \{s_2\}$. For each $s \in S$, let $q_s = 1$. Let $C = \{c_1, c_2\}$. Suppose that $c_1 \succ_{s_1} c_2$

and $c_1 \succ_{s_2} c_2$.

Suppose that φ is fair and separable but never picks a wasteful match. Then there are φ^1 and φ^2 such that for each $P \in \mathcal{P}^C$, $\varphi(P) = \mathcal{C}(\{\varphi^1(P^1), \varphi^2(P^2)\}, P)$.

Consider the $P \in \mathcal{P}^C$ such that $s_1\ P_1\ s_2\ P_1\ \varnothing$ and $s_1\ P_2\ s_2\ P_2\ \varnothing$. Since φ is does not pick wasteful matches, there are only two cases:

Case 1: $\varphi(P)(c_1) = s_1$. Since φ is does not pick wasteful matches, $\varphi(P)(c_2) = s_2$. Thus, $\varphi^2(P^2)(c_2) = s_2$.[25] Let $P' \in \mathcal{P}^C$ be such that $s_2\ P'_1\ s_1\ P'_1\ \varnothing$ and $s_2\ P'_2\ s_1\ P'_2\ \varnothing$. Since $P^2 = P'^2$, $\varphi^2(P'^2)(c_2) = s_2$. Thus $\varphi(P')(c_2) = s_2$ even though $s_2\ P'_1\ s_1\ P'_1\ \varnothing$ and $c_1 \succ_{s_2} c_2$. Thus $\varphi(P')$ is not fair at P'.

Case 2: $\varphi(P)(c_1) = s_2$. Since φ is does not pick wasteful matches, $\varphi(P)(c_2) = s_1$ even though $s_1\ P_1\ s_2$ and $c_1 \succ_{s_1} c_2$. Thus $\varphi(P)$ is not fair at P. □

A.2 Proof of Theorem 2

Before proving Theorem 2, we prove a Lemma that the sup of two fair matches is not only a match, but is fair itself.

Lemma 1. *For each $P \in \mathcal{P}$ and each pair $\nu, \nu' \in F(P)$, $\mathcal{C}(\{\nu, \nu'\}, P) \in F(P)$.*

Proof: Let $\mu = \mathcal{C}(\{\nu, \nu'\}, P)$. Suppose that $\mu \notin \mathcal{M}$. Then there is $s \in S$ such that $|\mu(s)| > q_s$. However, since $\nu, \nu' \in M$, $|\nu(s)| \leq q_S$ and $|\nu'(s)| \leq q_s$.

Let $c \in \mu(s)$ have the lowest priority at s among $\mu(s)$. That is, for each $c' \in \mu(s)$, $c' \succeq_s c$. By definition of μ, either $\nu(c) = s$ or $\nu'(c) = s$.

Case 1: $\nu(c) = \nu'(c) = s$. Since $|\nu(s)| \leq q_s$ and $|\nu'(s)| \leq q_s$, there is $c' \in \mu(s) \setminus \{c\}$ such that either $\nu(c') \neq s$ or $\nu'(c') \neq s$. Suppose that $\nu(c') \neq s$. By fairness of ν, since $\nu(c) = s$ and $c' \succ_s c$, $\nu(c')\ P_{c'}\ s = \mu(c')$. This contradicts the definition of μ. The proof is analogous if $\nu'(c') \neq s$.

Case 2: $\nu(c) = s$ and $\nu'(c) \neq s$. Then there is $c' \in \mu(c) \setminus \{c\}$ such that $\nu(c') \neq s$. By fairness of ν, since $\nu(c) = s$ and $c' \succ_s c$, $\nu(c')\ P_{c'}\ s = \mu(c')$. This contradicts the definition of μ.

Case 3: $\nu'(c) = s$ and $\nu(c) \neq s$. This case is analogous to Case 2.

[25]Recall that P^2 is P restricted to S^2 and that $\varphi^2(P^2)$ is the child optimal stable match for the problem $(P^2, \succ_{S^2})$.

Thus, we have shown that $\mu \in \mathcal{M}$. Now suppose there are $c \in C$ and $s \in S$ such that $s \; P_c \; \mu(c)$ and there is $c' \in \mu(s)$ for whom $c \succ_s c'$. Then, $s \; P_c \; \nu(c)$ and $s \; P_c \; \nu'(c)$. However, since $\mu(c') = s$, either $\nu(c') = s$ or $\nu(c') = s$. If $\nu(c') = s$ then ν is not fair and if $\nu'(c') = s$ then ν' is not fair. This contradicts the assumption that ν and ν' are fair. □

Proof of Theorem 2: This follows from Lemma 1. If $\nu, \nu' \in F(P)$, and $\nu \neq \nu'$ then $\mu = \mathcal{C}(\{\nu, \nu'\}, P)$ Pareto-dominates both ν and ν' and $\mu \in F(P)$. Since $F(P)$ is finite, there is a Pareto-best element of $F(P)$. □

A.3 Proofs of Propositions 1 and 2 and Theorem 3

We start with two lemmas. The first says that if a student is matched in the $(l-1)^{\text{th}}$ iteration then he is matched in the l^{th} iteration.

Lemma 2. *In the definition of $\varphi^{L\text{-}CO}$, if for $c \in C$ and $l = 1, \ldots, L$, if there is $k \in K$ such that $\mu_{l-1}(c) \in S^k$, then $\mu_l^k(c) \neq \varnothing$.*

Proof: Let $c \in \mathcal{C}$. Suppose that $k \in K$ is such that $\mu_{l-1}(c) \in S^k$. For each $c' \in C$, $U((P_{c'}^l)^k, \varnothing) \subseteq U((P_{c'}^{l-1})^k, \varnothing)$. Further, $\mu_{l-1}(c) \; (P_c^l)^k \; \varnothing$. By Theorem 2 of Gale and Sotomayor (1985), $\mu_l^k(c) \; P_c \; \mu_{l-1}^k(c) = \mu_{l-1}(c) \; P_c \; \varnothing$. □

The next lemma says that the sup of a set of fair matches is itself a fair match. It follows from Lemma 1

Lemma 3. *For each $P \in \mathcal{P}^C$ and each $M \subseteq F(P)$, $\mathcal{C}(M, P) \in F(P)$.*

Proof: Let $\mu_1, \ldots \mu_{|M|}$ be a labeling of M. Define, for each $l = 1, \ldots, |M| - 1$, $\nu_l : C \to S \cup \{\varnothing\}$ as follows.

$$\begin{aligned} \nu_1 &= \mathcal{C}(\{\mu_1, \mu_2\}, P) \text{ and} \\ \text{for } l = 2, \ldots, |M| - 1, \nu_l &= \mathcal{C}(\{\nu_{l-1}, \mu_{l+1}\}, P). \end{aligned}$$

By Lemma 1, since μ_1 and μ_2 are fair, $\nu_1 \in F(P)$. For each $l \geq 2$, by the same argument, $\nu_l \in F(P)$.

Since, $\nu_{|M|-1} = \mathcal{C}(M, P)$ the proof is complete. □

Proof of Proposition 1: That $\varphi^{L\text{-}CO}$ is individually rational is obvious. We prove that it is fair by showing that for each $l = 1, \ldots, L$, μ_l from the definition of $\varphi^{L\text{-}CO}$ is fair. We show this by induction over l.

By definition, for each $k \in K$, $\mu_1^k \in F(P)$. By Lemma 3, μ^1 is fair.

As an induction hypothesis suppose that μ_{l-1} is fair for some $l \geq 2$. We complete the proof by proving the induction step that μ_l is fair.

If μ_l is not fair, there are $s \in S$ and a pair $c, c' \in C$ such that $s\ P_c\ \mu_l(c)$, $\mu_l(c') = s$, and $c \succ_s c'$. Let $k \in K$ be such that $s \in S^k$. Then, since μ_l^k is fair at $(P^l)^k$ it follows that $\mu_l^k(c) = \varnothing$ so that $\varnothing\ (P_c^l)^k\ s$. This implies that $\mu_{l-1}(c)\ P_c\ s$. Then, there is $j \in K \setminus \{k\}$ such that $\mu_{l-1}(c) \in S^j$. By Lemma 2, $\mu_l^j(c) \neq \varnothing$. So $\mu_l(c)\ R_c\ \mu_l^j(c)\ R_c\ \mu_{l-1}^j(c) = \mu_{l-1}(c)\ P_c\ s$. Thus, $\mu_l(c)\ P_c\ s$. This contradicts the assertion that $s\ P_c\ \mu_l(c)$. □

Proof of Proposition 2: To show this, we prove that, for each $P \in \mathcal{P}^C$, in the definition of $\varphi^{(L+1)\text{-}CO}(P)$, for each $l = 1, \ldots, L$, μ_{l+1} Pareto-dominates μ_l.

Suppose there is $c \in C$ such that $\mu_l(c)\ P_c\ \mu_{l+1}(c)$. For each $k \in K$, by individual rationality of ψ^{kCO}, μ_{l+1}^k is individually rational at $(P^k)^{l+1}$. It then follows that $\mu_{l+1}^k(c) \in \{\varnothing\} \cup U(P_c, \mu_l(c))$. If $\mu_{l+1}(c) = \varnothing$ then for each $k \in K$, $\mu_{l+1}^k(c) = \varnothing$. By Lemma 2, $\mu_l(c) = \varnothing$. This contradicts the assumption that $\mu_l(c)\ P_c\ \mu_{l+1}(c)$.

If $\mu_{l+1}(c) \neq \varnothing$ then there is $k \in K$ such that $\mu_{l+1}(c) = \mu_{l+1}^k(c) \in S^k$. Since $\mu_{l+1}(c) \in S^k$ and $\mu_{l+1}^k(c) \in \{\varnothing\} \cup U(P_c, \mu_l(c))$, it follows that $\mu_{l+1}(c)\ R_c\ \mu_l(c)$. This too contradicts the assumption that $\mu_l(c)\ P_c\ \mu_{l+1}(c)$. □

Proof of Theorem 3: In the definition of $\varphi^{L\text{-}CO}(P)$ if for some $l = 2, \ldots, L-1$, $\mu_{l-1} = \mu_l$, then for every $l' > l, \mu_{l'} = \mu_l$ since for each $k \in K$, $(P^l)^k = (P^{l'})^k$. Since $\mathcal{M}$ is a finite set, we need only show that if $\mu_{l-1} = \mu_l$ then μ_l is not wasteful.

We start by showing that for each $c \in C$ there is at most one $k \in K$ such that $\mu_l^k(c) \neq \varnothing$. If $\mu_l(c) = \varnothing$ then, by Lemma 2, for each $k \in K$, $\mu_l^k(c) = \varnothing$. If, for some $k \in K$, $\mu_{l-1}(c) = \mu_l(c) = s \in S^k$ then $\mu_{l-1}^k(c) = \mu_l^k(c) = s$ and for each $j \in K \setminus \{k\}$, $\mu_l^j(c) \in U(P_c, s) \cup \{\varnothing\}$. Since $\mu_l(c) = s$ it follows that $s\ R_c\ \mu_l^k(c)$. Thus $\mu_l^j(c) = \varnothing$.

So for each $k \in K$, and each $s \in S^k$, $\mu_l(s) = \mu_l^k(s)$.

Suppose that μ_l is wasteful. Then there are $c \in C$ and $s \in S$ such that $s \; P_c \; \mu_l(c)$ and $|\mu_l(s)| < q_s$. Let $k \in K$ be such that $s \in S^k$.

Case 1: $\mu_l(c) \in S^k \cup \{\varnothing\}$. Then $\mu_l^k(c) = \mu_l(c)$ and so μ_l^k is wasteful at $(P^l)^k$.

Case 2: For some $j \in K \setminus \{k\}$, $\mu_l(c) \in S^j$. Then $\mu_l^k(c) = \varnothing$ but $s \; (P_c^l)^k \; \varnothing$. Thus μ_l^k is wasteful at $(P^l)^k$.

By definition of ψ^{kCO}, μ_l^k is not wasteful at $(P^l)^k$. This contradiction shows that μ_l is not wasteful. □

A.4 Proofs of Propositions 4, 5, and 6 and Theorem 4

We find the following notion useful in the proofs. For each $s \in S$, each $c \in C$, given $\kappa \in [0,1]^S$, we say that child *c vies for a seat at s* if for each $s' \in S$ such that $s' \; P_c \; s$, $p_{s'c} > \kappa_{s'}$. That is, c does not have a high enough priority to be matched to any school that he prefers to s.

Proof of Proposition 4: Let $\mu = \varphi^{CO}(\rho)$. Since, for each $s \in S \cup \{\varnothing\}$, $s_1 \; R_0 \; s$, and since μ is fair, non-wasteful, and individually rational, $\rho(\mu(s_1)) = q_{s_1} = \frac{1}{2m}$. Furthermore, since for every $c \in C$ and $s \in S, s_1 \; R_c \; s$, there is a mass $\rho(C) = 1$ of children who vie for q_{s_1} seats. Since priorities are uniformly distributed, the mass of such $c \in C$ with $p_c \leq \kappa_{s_1}$ is $\kappa_{s_1}\rho(C)$. Setting $\kappa_{s_1}\rho(C) = q_{s_1}$, we deduce that

$$\kappa_{s_1} = \frac{1}{\rho(C)} q_{s_1} = \frac{1}{1}\frac{1}{2m} = \frac{1}{2m}.$$

So $\mu(s_1) = \{c \in C : p_{s_1 c} \leq \frac{1}{2m}\}$.

Since, for each $s \in S \cup \{\varnothing\} \setminus \{s_1\}$, and for each $c \in C \setminus \mu(s_1)$, $s_2 \; R_c \; s$, there is a mass $\rho(C \setminus \mu(s_1))$ of children who vie for q_{s_2} seats. Thus, we deduce that

$$\kappa_{s_2} = \frac{1}{\rho(C \setminus \mu(s_1))} q_{s_2} = \frac{1}{(1 - \frac{1}{2m})} \frac{1}{2m} = \frac{1}{2m - 1}$$

and $\mu(s_2) = \{c \in C : p_{s_2 c} \leq \frac{1}{2m-1}\}$.

Continuing this argument, for each $k \in \{3, \ldots 2m\}$,

$$\kappa_{s_k} = \frac{1}{\rho(C \setminus (\mu(s_1) \cup \cdots \cup \mu(s_{k-1})))} q_{s_k} = \frac{1}{(1 - \frac{k-1}{2m})} \frac{1}{2m} = \frac{1}{2m - k + 1}$$

and $\mu(s_k) = \{c \in C : p_{s_k c} \leq \frac{1}{2m-k+1}\}$.

Since $\kappa_{s_{2m}} = 1$, no child is left unmatched. $\square$

Proof of Proposition 5: The first step is to show that the cutoffs of μ_1^1 and μ_1^2 are the ones in the statement of the proposition. The proof is identical to that of Proposition 4.

Next, we note that the cutoffs of μ_1 for schools in S^1 are those of μ_1^1. Similarly for schools in S^2.

To see that $\frac{1}{4}$ of students are unmatched, note that for each $k \in \{1, \ldots, m\}$, $(1 - \kappa_{s_{2k-1}}) = (1 - \kappa_{s_{2k}}) = \frac{2m-k}{2m-k+1}$. Thus, by the assumption that priorities are independently and uniformly drawn,

$$\rho(\Upsilon_\kappa) = \left(\frac{2m-1}{2m}\right)\left(\frac{2m-1}{2m}\right)\left(\frac{2m-2}{2m-1}\right)\left(\frac{2m-2}{2m-1}\right)\cdots\left(\frac{m}{m+1}\right)\left(\frac{m}{m+1}\right)$$

So, $\rho(\Upsilon_\kappa) = \frac{1}{4}$. $\square$

Proof of Proposition 6: We start by noting that P^2 in the definition of $\varphi^{2\text{-CO}}$ is determined by truncating P at $\mu = \varphi^{1\text{-CO}}(\rho)$. For each $s \in S$, let κ_s^1 be the cutoff for s of $\mu_1 = \varphi^{1\text{-CO}}(\rho)$.

For each $k \in \{1, \ldots 2m\}$, we will denote by v_k^2 the set of children who vie for a seat at s_k in the second iteration. By our assumption that priorities are independently and uniformly distributed, $\kappa_{s_k}^2 = q_{s_k} \frac{1}{\rho(v_k^2)} = \frac{1}{2m\rho(v_k^2)}$.

We start with schools in S^1. In particular, we begin with s_1. As before, v_1^2 is C since at $(P^2)^1$ every child ranks s_1 above every other school in S^1. Consequently, $\kappa_{s_1}^2 = \frac{1}{2m}$.

Next, we consider s_3. Every child except those admitted to s_1 and those admitted to s_2 by μ_1 vies for a seat at s_3. That is, $v_3^2 = \Delta(\kappa_{s_1}^2) \cap \Delta(\kappa_{s_2}^1)$. Since priorities are uniformly distributed, we have $\rho(v_3^2) = (1 - \kappa_{s_1}^2)(1 - \kappa_{s_2}^1)$. Thus, $\rho(v_3^2) = \left(\frac{2m-1}{2m}\right)\left(1 - \frac{1}{2m}\right)$. So

$$\kappa_3^2 = \frac{1}{(2m-1)\left(1 - \frac{1}{2m}\right)}.$$

For general $k \in \{2, \ldots, m\}$, we prove that $\kappa_{s_{2k-1}}^2$ takes the value in the statement of the proposition by induction on k. The set of children who vie for a seat at s_{2k-1} are those who vie for a seat at s_{2k-3} but are denied not only at s_{2k-3} at cutoff $\kappa^2_{s_{2k-3}}$ but also at s_{2k-2} at cutoff $\kappa^1_{s_{2k-2}}$. That is,

$$v^2_{2k-1} = v^2_{2k-3} \cap \Delta(\kappa^2_{s_{2k-3}}) \cap \Delta(\kappa^1_{s_{2k-2}}).$$

Thus, $\rho(v^2_{sk-1}) = \rho(v^2_{2k-3})(1 - \kappa^2_{s_{2k-3}})(1 - \kappa^1_{s\,2k-2})$. That is,

$$\rho(v^2_{2k-1}) = \rho(v^2_{2k-3}) \left(1 - \frac{1}{2m\rho(v^2_{2k-3})}\right) \left(\frac{2m-k-1}{2m-k+2}\right).$$

By the induction hypothesis, $\rho(v^2_{2k-3}) = \frac{2m-(k-2)}{2m} \left(1 - \sum_{j=2m-(k-3)}^{2m} \frac{1}{j}\right)$. Combining these, we have

$$\rho(v^2_{2k-1}) = \frac{2m-k+1}{2m} \left(1 - \sum_{j=2m-(k-2)}^{2m} \frac{1}{j}\right).$$

Since $\kappa^2_{2k-1} = \frac{1}{2m\rho(v^2_{2k-1})}$ we have shown that it takes the desired value.

We now proceed to S^2. Every child, except those admitted to s_1 by μ_1, vies for s_2. Thus, $v^2_2 = \Delta(\kappa^1_{s_1})$. Since $\rho(v^2_2) = 1 - \frac{1}{2m}$ we have $\kappa^2_2 = \frac{1}{2m-1}$.

For $k \in \{2, \ldots, m\}$ we show by induction that κ^2_{2k} takes the desired value by induction on k. Similar to the case of s_{2k-1},

$$v^2_{2k} = v^2_{2k-2} \cap \Delta(\kappa^2_{s_{2k-2}}) \cap \Delta(\kappa^1_{s_{2k-1}}).$$

Thus,

$$\rho(v^2_{2k}) = \rho(v^2_{2k-2}) \left(1 - \frac{1}{2m\rho(v^2_{2k-2})}\right) \left(\frac{2m-k}{2m-k+1}\right).$$

By the induction hypothesis, $\rho(v^2_{2k-2}) = \frac{2m-(k-1)}{2m}\left(1 - \sum_{j=2m-(k-2)}^{2m-1} \frac{1}{j}\right)$. Thus,

$$\rho(v^2_{2k}) = \frac{2m-k}{2m}\left(1 - \sum_{j=2m-(k-1)}^{2m-1} \frac{1}{j}\right).$$

Since $\kappa^2_{2k} = \frac{1}{2m\rho(v^2_{2k})}$ we have shown that it takes the desired value. □

Proof of Theorem 4: We first show that for each $k \in \{3, \ldots, 2m\}$, $\kappa^L_{s_k} = \frac{\kappa^L_{k-2}}{(1-\kappa^L_{k-2})(1-\kappa^{L-1}_{k-1})}$. By the same argument as in the proof of Proposition 6,

$$v^L_k = v^L_{k-2} \cap \Delta(\kappa^L_{s_{k-2}}) \cap \Delta(\kappa^{L-1}_{s_{k-1}})$$

and $\rho(v^L_k) = \rho(v^L_{k-2})(1-\kappa^L_{s_{k-2}})(1-\kappa^{L-1}_{k-1})$. Since $\rho(v^L_{k-2}) = \frac{1}{2m\kappa^L_{k-2}}$,

$$\rho(v^L_k) = \frac{(1-\kappa^L_{s_{k-2}})(1-\kappa^{L-1}_{k-1})}{2m\kappa^L_{s_{k-2}}}$$

Since $\kappa^L_{s_k} = \frac{1}{2m\rho(v^L_k)}$, we have the desired conclusion.

Next, we show that for each $k \in \{1, \ldots, L\}$, $\kappa^L_k = \frac{1}{2m-k+1}$. Proposition 6 shows that this holds for $L = 2$. Let $L > 2$. As an induction hypothesis, suppose that this holds for each $L' < L$. Thus, for each $k \in \{1, \ldots, L-1\}$, $\kappa^{L-1}_{s_k} = \frac{1}{2m-k+1}$.

By definition of P^L, for each $c \in C$, and each $s' \in S \cup \{\varnothing\} \setminus \{s_1\}$, $s_1 \; P^L_c \; s'$. Thus, $v^L_1 = C$ and so $\kappa^L_{s_1} = \frac{1}{2m}$. Similarly, for each $c \in C \setminus \mu_L(s_1)$ and each $s' \in S \cup \{\varnothing\} \setminus \{s_1, s_2\}$, $s_1 \; P^L_c \; s_2 \; P^L_c \; s'$. Thus, $v^L_2 = C \setminus \mu_L(s_1)$. So $\kappa^L_{s_2} = \frac{1}{2m(1-\kappa^L_{s_1})} = \frac{1}{2m\left(1-\frac{1}{2m}\right)} = \frac{1}{2m-1}$.

For $k \in \{3, \ldots, L\}$, assuming as induction hypotheses that $\kappa^L_{s_{k-2}} = \frac{1}{2m-(k-2)+1}$

and $\kappa^L_{s_{k-1}} = \frac{1}{2m-(k-1)+1}$,

$$\begin{aligned}
\kappa^L_{s_k} &= \frac{\kappa^L_{s_{k-2}}}{(1-\kappa^L_{s_{k-2}})(1-\kappa^L_{s_{k-1}})} \\
&= \frac{\frac{1}{2m-(k-2)+1}}{\left(1-\frac{1}{2m-(k-2)+1}\right)\left(1-\frac{1}{2m-(k-1)+1}\right)} \\
&= \frac{\frac{1}{2m-k+3}}{\left(\frac{2m-k+2}{2m-k+3}\right)\left(\frac{2m-k+1}{2m-k+2}\right)} \\
&= \frac{1}{2m-k+1}.
\end{aligned}$$

□

A.5 Proof of Proposition 7 and Theorem 5

Proof of Proposition 7: By the assumption that preferences are uniformly distributed, for each $s \in S$, there are $\frac{1}{2m}$ children who rank it first among all schools. It is fair, individually rational, and non-wasteful to match each child to his most preferred school. This is supported by κ such that for each $s \in S$, $\kappa_s = 1$. Since this is the only fair, individually rational, and non-wasteful match, these are the cutoffs at $\varphi^{CO}(\rho)$. □

Proof of Theorem 5: In the context of calculating the cutoff for a particular $s \in S$, we will say "this group" to mean the group of schools that s is in and "the other group" to mean the group that s is not in.

Because of the symmetry of the problem, every school will have the same cutoff. So for each $l \in \{1, \dots, L\}$, we denote by κ_l the common cutoff associated with μ_l.

Since we will induct on l, as a base case we start with κ_1. To calculate κ_1, preferences over the other group do not matter. Let $s \in S$ and let S^i be this group. We consider the set of children, ν^1_s, who vie for s at the cutoffs $(\kappa_{1S^i}, 0)$ where each school in S^i has the same cutoff κ_1 and each school not in S^i has the cutoff 0.

By the assumption that preferences are uniformly distributed, for each $t \in \{1, \dots, m\}$, there is a mass $\frac{1}{m}$ of children who rank s t^{th} among S^i.

Every child who ranks s first vies for it. There is a mass $\frac{1}{m}$ of such children. Every child who ranks s second vies for it if and only if he is denied by his top choice. By the assumption that preferences and priorities are uniformly distributed, there is a mass $(1-\kappa_1)\frac{1}{m}$ of such children. In general, for each $t \in \{1, \dots, m\}$ every child who ranks s t^{th} vies for it if and only if he is denied by each of the $t-1$ schools that he ranks above s. There is a mass $(1-\kappa_1)^{t-1}\frac{1}{m}$ of such children. In total, the mass of children who vie for s is

$$\begin{aligned} \rho(\nu_s^1) &= \tfrac{1}{m} + (1-\kappa_1)\tfrac{1}{m} + \cdots + (1-\kappa_1)^{m-1}\tfrac{1}{m} \\ &= \tfrac{1-(1-\kappa_1)^m}{m\kappa_1} \end{aligned}$$

By the assumption that priorities are uniformly drawn, κ_1 satisfies the following condition:

$$\kappa_1 \rho(\nu_s^1) = q_s = \frac{1}{2m}.$$

Substituting $\rho(\nu_s^1)$ in this equation,

$$\kappa_1 \frac{1-(1-\kappa_1)^m}{m\kappa_1} = \frac{1}{2m}.$$

Solving for κ_1 yields $\kappa_1 = 1 - \frac{1}{2^{\frac{1}{m}}}$.

As an induction hypothesis, suppose that $\kappa_{l-1} = 1 - \frac{1}{l^{\frac{1}{m}}}$. We next show that that $\kappa_l = 1 - \frac{1}{(l+1)^{\frac{1}{m}}}$.

We consider the set of children, ν_s^l, who vie for s at the cutoffs $(\kappa_{lS^i}, \kappa_{l-1S^j})$: schools in this group, S^i, are associated with cutoffs κ_l and those in the other group, S^j, are associated with κ_{l-1}.

First, consider the children denied by every school in S^j. There is a mass $(1 - \kappa_{l-1})^m$ of such children. By the induction hypothesis, $(1-\kappa_{l-1})^m = \left(\frac{1}{l^{\frac{1}{m}}}\right)^m = \frac{1}{l}$. To determine what proportion of these children vie for s we apply an argument identical to the one used to calculate κ_1. That preferences are truncated does not change the argument since if a child's preferences are truncated at the t^{th}

position, then that child is *not* denied by at least one school among his top t schools. Thus, among the $\frac{1}{l}$ mass of children denied by every school in S^j at κ_{l-1}, a mass $\frac{1}{l}\frac{1-(1-\kappa_l)^m}{m\kappa_l}$ of children vies for s.

There is a mass $1-\frac{1}{l}$ of children not denied by at least one school in S^j at κ_{l-1}. The preferences of each of these children is truncated. If preferences were not truncated, the proportion of these children who would vie for s, by the above argument, would be $\frac{1-(1-\kappa_l)^m}{m\kappa_l}$. By symmetry, half of these children's preferences are truncated above s and half below. Consequently only half of them vie for s with truncated preferences. Thus, of the children who are not denied by at least one school in S^j at κ_{l-1}, the proportion of children who vie for s is $\frac{1}{2}\frac{1-(1-\kappa_l)^m}{m\kappa_l}$. From the uniformity assumption, the total mass of such children who vie for s is $(1-\frac{1}{l})\frac{1}{2}\frac{1-(1-\kappa_l)^m}{m\kappa_l} = \frac{l-1}{2l}\frac{1-(1-\kappa_l)^m}{m\kappa_l}$.

Adding these two masses together,

$$\rho(\nu_s^l) = \frac{1}{l}\frac{1-(1-\kappa_l)^m}{m\kappa_l} + \frac{l-1}{2l}\frac{1-(1-\kappa_l)^m}{m\kappa_l} = \frac{l+1}{2l}\left(\frac{1-(1-\kappa_l)^m}{m\kappa_l}\right).$$

By the assumption that priorities are uniformly drawn, κ_l satisfies the following condition:

$$\kappa_l\rho(\nu_s^l) = q_s = \frac{1}{2m}.$$

Substituting $\rho(\nu_s^l)$ in this equation,

$$\kappa_l\left[\frac{l+1}{2l}\left(\frac{1-(1-\kappa_l)^m}{m\kappa_l}\right)\right] = \frac{1}{2m}.$$

$$\text{So} \qquad 1-(1-\kappa_l)^m = \frac{l}{l+1}.$$

Thus, $\kappa_l = 1-\frac{1}{(l+1)^{\frac{1}{m}}}$.

The mass of children left unmatched by μ_l is just $(1-\kappa_l)^{2m} = \frac{1}{(l+1)^2}$. □

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