The difference indifference makes in strategy-proof allocation of objects*

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Abstract

We study problems of allocating objects among people. Some objects may be initially owned and the rest are unowned. Each person needs exactly one object and initially owns at most one object. We drop the common assumption of strict preferences. Without this assumption, it suffices to study problems where each person initially owns an object and every object is owned. For such problems, when preferences are strict, the "top trading cycles" algorithm provides the only rule that is efficient, strategy-proof, and individually rational [1]. Our contribution is to generalize this algorithm to accommodate indifference without compromising on efficiency and incentives.

JEL classification: C71, C78, D71, D78

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1 Introduction

We study problems of allocating objects to people. In particular, we are interested in problems where each person needs a single object and initially owns at most one object. Furthermore, these problems do not involve other goods (divisible or indivisible) that can be allocated along with the objects. Think of the allocation of seminar slots, on-campus apartments, or organs for transplant. This is a broad class of problems including, at the extremes, those where no one initially owns an object [2] and those where everyone initially owns an object [3]. We account for people being indifferent between objects. Consequently, as we discuss in Section 3, it turns out that it suffices to consider problems where everyone initially owns an object and every object is owned.

Most of the related literature deals only with problems where no one is ever indifferent. We argue that there are many real-world situations where preferences do exhibit indifference. For instance, if preferences are based on coarse descriptions (say, from a housing brochure), there may be insufficient information to break ties. Alternatively, if preferences are based on checklists of criteria (like blood and tissue types for organ transplant), distinct objects satisfying exactly the same criteria are equivalent. Appropriate design of rules should take indifferences into account since breaking ties arbitrarily may lead to inefficiencies.

Our contribution is to define a class of rules that are *strategy-proof*, *Pareto-efficient*, and *individually rational* for such problems. Our rules are based on a new algorithm. Alcalde-Unzu and Molis [4] have simultaneously and independently proposed another such algorithm. The compatibility of these three properties had been an open question until these works. In fact, an earlier result [5] is that these three properties are incompatible with *non-bossiness*.

Our algorithm is a novel adaptation of Gale's "top trading cycles" algorithm [3]. The *top trading cycles* algorithm, which is defined only when there are no indifferences, proceeds by iterating the following: Each person "points" at the person who owns his most preferred object. Since each person points, there is at least one "top cycle:" a group of people, each of whom has the next person's most preferred object and the last of whom has the first person's most preferred object. The algorithm assigns his most preferred object to each member of such a cycle and removes him from the problem. This continues until no one is left.

There are two questions to answer in generalizing this algorithm to deal with indifference. First, when can people be removed from the problem? We formulate a condition that is akin to equating demand and supply. Once a group of people trades (as a part of a cycle), we check if what they hold (think of this as supply) is exactly the set of objects that they most prefer (think of this as demand). If not, they may be able to improve the welfare of someone outside of the group and we keep them in the problem.

Second, who should a person point at if he is indifferent between what two different people hold? To answer this, we first note that only a person who does not hold one of his most preferred objects can be made any better off than he already is. Such a person is "unsatisfied." The rest of the people are "satisfied." When a person is directly or indirectly pointed at, this increases the possible trades available to him. The objective is to increase the trades available to *unsatisfied* people. We define a recursive way of determining whom each person points at. First, anyone who can point at an *unsatisfied* person does so, breaking ties with a fixed priority order. Next, anyone who can point at an *unsatisfied* person *through* a *satisfied* person does so. These people break ties with the same fixed priority order over the eventual *unsatisfied* person that they would reach. Then, we consider anyone who can point at an *unsatisfied* person through two *satisfied* people, and so on.

The use of a priority order means that one person may be systematically favored over another. However, this is what allows us to achieve *strategy-proofness*.

These changes add considerably to the complexity of the algorithm. This stands in stark contrast with the simplicity of the *top trading cycles* algorithm in the absence of indifferences. When people are never indifferent between objects, the *core* consists of a unique allocation which is also the unique *competitive allocation* [6]. This is the allocation identified by the *top trading cycles* algorithm. The rule that maps each problem with its unique core allocation is not only *strategy-proof* [7] but also *group strategy-proof* [8]. Further, it is the only *strategy-proof, Pareto-efficient,* and *individually rational* rule [1, 9]. It is also *non-bossy* and *anonymous* [10].

The difficulties in accounting for indifference are highlighted by the fact that these results no longer hold in the presence of indifference. A *core* allocation is not guaranteed to exist [3].[1] Further, the set of *competitive allocations* no longer coincides with the *core* [12].[2] [3] *Group strategy-proofness* and *Pareto-efficiency* are incompatible [13].

The bulk of our efforts in dealing with indifference are in attaining *Pareto-efficiency* rather than settling for *weak Pareto-efficiency*. If we are satisfied with just *weak Pareto-efficiency*, which is particularly weak in this context, then breaking ties in an arbitrary, but fixed, way and applying the *top trading cycles* algorithm provides a *strategy-proof, weakly Pareto-efficient* and *individually rational* rule. In fact, this is the only way to achieve these three properties along with *non-bossiness* and a certain version of *consistency* [14].

[1]Quint and Wako [11] provide necessary and sufficient conditions on preference profiles for the core to be non-empty.

[2]Yet, the *weak core* is nonempty [3] and our rules select from it.

[3]In fact, we show (Example 1) that for some profiles of preferences, there may not even exist a *competitive allocation* that is *Pareto-efficient*

The remainder of the paper is organized as follows. We present the model where every object is owned and each person owns an object in Section 2. In Section 3 we show that it is without loss of generality to study such a model since it includes cases where some people do not own an object and some objects are not owned. We describe some desiderata of allocations and rules in Section 4 and define our rules in Section 5. In Section 6, we present our results.

2 The Model

Let $\boldsymbol{O}$ be a set of distinct objects. Let $\boldsymbol{N}$ be a set of people. There are exactly as many objects as people: $|O| = |N|$. An **endowment** is a bijection, $\omega : N \to O$, that associates an object with each person. For each $i \in N$, i's component of the endowment is $\omega(i)$. Each person has a preference relation over O. Let the set of all preference relations be $\mathcal{R}$. A **preference profile** associates each individual with a preference relation in $\mathcal{R}$. Let $\mathcal{R}^N$ be the set of all preference profiles. Given a profile $R \in \mathcal{R}^N$, for each $i \in N$, i's preference relation is R_i. For each pair of alternatives, $a, b \in O$, if i finds a to be at least as good as b, we write $a \; R_i \; b$. If a is better than b, that is, $a \; R_i \; b$ but not $b \; R_i \; a$, we write $a \; P_i \; b$. Similarly, if i is indifferent between a and b, we write $a \; I_i \; b$. Let $\mathcal{P} \subset \mathcal{R}$ be the set of "strict" preference relations. That is, $\mathcal{P} \equiv \{R_0 \in \mathcal{R} : \text{ for each pair } a, b \in O, a \; I_0 \; b \Leftrightarrow a = b\}$.

We use the notation R_{-i} to denote the preference relations of everyone but i. For each group $S \subseteq N$, we denote the preferences of all the people in S by R_S, and those not in S by R_{-S}. We denote the set of all preferences for people in the group S by $\mathcal{R}^S$.

Let $\boldsymbol{A}$, the set of all bijections from N to O, be the set of all possible allocations. For each $\alpha \in A$, and each $i \in N$, let $\alpha(i)$ denote i's component of α. Similarly, for each $S \subseteq N$, let $\alpha(S)$ be the collective assignment to members of S under α. That is, $\alpha(S) = \bigcup_{i \in S}\{\alpha(i)\}$.

A **problem** consists of a preference profile and an endowment, $(R, \omega) \in \mathcal{R}^N \times A$. A **rule**, $\varphi : \mathcal{R}^N \times A \to A$, selects an allocation for each problem.

3 Generality of our model

In this section, we show that the model that we have studied is general enough to include problems where there may or may not be a private endowment in addition to a social endowment [2, 15].

Let $\tilde{O}$ be a set of objects and $\tilde{N}$ be a set of people. Let $\varnothing \notin \tilde{O}$ be the **null object**. The **private endowment,** $\tilde{\boldsymbol{\omega}} : \tilde{\boldsymbol{N}} \to \tilde{\boldsymbol{O}} \cup \{\varnothing\}$, is such that for each $i, j \in \tilde{N}$, $\tilde{\omega}(i) \neq \tilde{\omega}(j)$ unless $\tilde{\omega}(i) = \varnothing$. Let $\tilde{\mathcal{R}}$ be the set of preference relations over $\tilde{O}$. Let $\tilde{R} \in \tilde{\mathcal{R}}^{\tilde{N}}$. The tuple $(\tilde{O}, \tilde{N}, \tilde{\omega}, \tilde{R})$ defines a problem. We show how this problem can be encoded as a problem in our original model without *social endowments*.

Define (O, N, ω, R) as follows. For each $a \in \tilde{O} \setminus \tilde{\omega}(\tilde{N})$, we introduce i_a, a "dummy person" with degenerate preferences, $R_{i_a} = \overline{I_0}$. For each $i \in \tilde{N}$ such that $\tilde{\omega}(i) = \varnothing$, we introduce d_i, a "dummy object" which every person considers to be worse than any object in $\tilde{O}$. For each person in $\tilde{N}$, his preferences over $\tilde{O}$ are kept the same. That is,

$$\begin{aligned} O &\equiv \tilde{O} \cup \{d_i : \text{for each } i \in \tilde{N} \text{ such that } \tilde{\omega}(i) = \varnothing\}, \\ N &\equiv \tilde{N} \cup \{i_a : \text{for each } a \in \tilde{O} \setminus \tilde{\omega}(\tilde{N})\}, \\ \text{For each } i \in N, \omega(i) &\equiv \begin{cases} \tilde{\omega}(i) & \text{if } i \in \tilde{N} \text{ and } \tilde{\omega}(i) \neq \varnothing \\ d_i & \text{if } i \in \tilde{N} \text{ and } \tilde{\omega}(i) = \varnothing \\ a & \text{if } i = i_a, \text{ and} \end{cases} \end{aligned}$$

$R \in \mathcal{R}^N$ is such that for each $i \in \tilde{N}$, $R_i|_{\tilde{O}} = \tilde{R}_i|_{\tilde{O}}$, and for each $d_j \in O \setminus \tilde{O}$ and each $a \in \tilde{O}, a \; P_i \; d_j$.

When preferences are strict and the problem includes private endowment, each *top cycles rule with fixed tie breaking* coincides with the *house for turn* rule associated with the same priority.[4] For the same domain of preferences with no public endowment, our family of rules collapses to the *core*. Moreover, when there is no private endowment, the *top cycles rule with a fixed tie breaking* according to some priority coincides with the *serial dictatorship* rule associated with the same priority.

We also point out that the *top cycles rules with fixed tie breaking* described in this paper can be generalized to problems in which each person may be endowed with any number of objects [16, 17].

For school choice problems, the objects that have to be allocated among students are seats at schools. These problems have three important characteristics: First, each seat at a school can be modeled as a copy of the same object. Second, students are indifferent between seats at the same school, but not between seats at different schools. Finally, schools have weak priorities over students. An adaptation of the *top cycles algorithm with fixed priority* can be used in this environment by treating each school's priority as an "inheritance hierarchy"[16, 18]. Though this would violate the schools' priorities, it would still result in an efficient and strategy-proof rule.

4 Properties of allocations and rules

In this section, we list some desiderata of allocations and rules. Let φ be a rule.

[4]This rule is known in the literature as the *"you request my house I get your turn"* rule [15].

The first requirement is that a rule respects each individual's endowment. That is, the allocation selected by the rule should not assign, to any person, an object that he finds worse than his endowment.

For each $(R, \omega) \in \mathcal{R}^N \times A$ and $\alpha \in A$, we say that **α is individually rational at (R, ω)** if for each $i \in N, \alpha(i)\ R_i\ \omega(i)$. Let **$IR(R, \omega)$** be the set of all *individually rational allocations at* (R, ω).

Individual Rationality: For each $(R, \omega) \in \mathcal{R}^N \times A$, $\varphi(R, \omega) \in IR(R, \omega)$.

Before we state the next requirement, we define an efficiency relation between allocations. For each $\alpha, \beta \in A$ and $R \in \mathcal{R}^N$, **α Pareto dominates β at R** if at least one person is better off at α than at β and nobody is worse off. That is, for some $i \in N$, $\alpha(i)\ P_i\ \beta(i)$ and for each $i \in N, \alpha(i)\ R_i\ \beta(i)$.

For each $R \in \mathcal{R}^N$, let the set of allocations that are not *Pareto dominated* by any other allocation be **$PE(R)$**.

Pareto-efficiency: For each $(R, \omega) \in \mathcal{R}^N \times A$, $\varphi(R, \omega) \in PE(R)$.

The next property says that unilaterally misreporting one's preferences is never beneficial.

Strategy-proofness: For each $(R, \omega) \in \mathcal{R}^N \times A$, there is no $i \in N$ for whom there is $R'_i \in \mathcal{R}$ such that

$$\varphi(\underbrace{R'_i}_{\text{lie}}, R_{-i}, \omega)(i) \underbrace{P_i}_{\text{truth}} \varphi(\underbrace{R_i}_{\text{truth}}, R_{-i}, \omega)(i).$$

The following is the requirement that nobody can affect what the rule assigns to others without affecting his own assignment.

Non-bossiness: For each $(R, \omega) \in \mathcal{R}^N \times$, there are no $i \in N$ and $R'_i \in \mathcal{R}$ such that

$$\varphi(R, \omega)(i) = \varphi(R'_i, R_{-i}, \omega)(i) \text{ and } \varphi(R, \omega) \neq \varphi(R'_i, R_{-i}, \omega).$$

The next desideratum is that the rule is a function of preferences and endowments, but not identities. Let $\pi : N \to N$ be a permutation of N. For each $(R, \omega) \in \mathcal{R}^N \times A$, define the **permutation of (R, ω) with respect to π,** as **(R^π, ω^π)** $\in \mathcal{R}^N \times A$, such that for each $i \in N$, $R^\pi_{\pi(i)} \equiv R_i$ and $\omega^\pi_{\pi(i)} \equiv \omega_i$.

Anonymity: For each $i, j \in N$, each $R \in \mathcal{R}^N$, $\omega \in A$, each $\pi : N \to N$, and each $i \in N$,

$$\varphi(R^\pi, \omega^\pi)(\pi(i)) = \varphi(R, \omega)(i).$$

The final requirement is that no group of people would rather re-allocate their endowments among themselves than participate in the application of the rule. This can be expressed in two ways. First, for each $\alpha \in A$, $(R, \omega) \in \mathcal{R}^N \times A$, and $S \subseteq N$, we say that **α is blocked by S** if members of S can re-allocate their endowments in a way that makes each of them better off than at α. That is, there is $\beta \in A$ such that $\beta(S) = \omega(S)$ and for each $i \in S, \beta(i)\ P_i\ \alpha(i)$. Second, we say that **$\alpha$ is weakly blocked by S** if members of S can re-allocate their endowments in a way that makes at least one of them better off than at α, without making any of the rest worse off than at α. That is, there is $\beta \in A$ such that $\beta(S) = \omega(S)$, for some $i \in S, \beta(i)\ P_i\ \alpha(i)$, and for each $i \in S, \beta(i)\ R_i\ \alpha(i)$.

The **weak core, $C^W(R, \omega)$**, is the set of allocations that are not *blocked* by any coalition and the **core, $C(R, \omega)$**, is the set of allocations that are not *weakly blocked* by any coalition.

5 Top cycles rules with fixed tie breaking

Let $\prec$ be a linear ordering of N.

The notion of "most preferred" objects among a subset of O is critical for the definition of our rule. For each $R \in \mathcal{R}^N$, $O' \subseteq O$, and $i \in N$, let **i's most preferred objects, under R_i, among O'**, be denoted by $\boldsymbol{\tau(R_i, O')} \equiv \{a \in A :$ for each $b \in O', a\ R_i\ b\}$.

Gale's "top trading cycles" algorithm [3] is applicable after breaking ties arbitrarily. The algorithm, defined for preferences in $\mathcal{P}^N$, proceeds in steps. Each step consists of three phases: First, in the "pointing phase," each person points at the person endowed with his most preferred object. Since each person points, there is at least one cycle. Next, in the "trading phase," the members of such cycles exchange their objects according to the way that they point. That is, if i and j are in a cycle and i points at j, then i receives $\omega(j)$. Finally, in the "departure phase," each person who has traded is removed with the object that he receives. The algorithm then continues to the next step.

The associated rules are *strategy-proof* and *individually rational* but not *Pareto-efficient*. To see this, consider the following example in which, no matter how ties are broken, the result of the "top trading cycles" algorithm is not *Pareto-efficient*.

Example 1. *Breaking ties.*

Let $N \equiv \{1, 2, 3\}$, $\omega \equiv (a, b, c)$, and $R \in \mathcal{R}^N$ be as follows.

R_1	R_2	R_3
$b\ c$	a	a
a	c	b
	b	c

There are exactly two ways to break ties: $P^1, P^2 \in \mathcal{P}^N$:

P^1_1	P^1_2	P^1_3
b	a	a
c	c	b
a	b	c

P^2_1	P^2_2	P^2_3
c	a	a
b	c	b
a	b	c

But for (P^1, ω) the recommendation is (b, a, c) and for (P^2, ω) it is (c, b, a). Neither of these is *Pareto-efficient*.[5] •

The next class of rules that we define are based on an adaptation of the *top trading cycles* algorithm.[6] We address two important questions in doing so:

1. When can a person be removed? In the previous example, if 2 gets a, 1 cannot leave with b since that would violate *Pareto-efficiency*. Clearly, it is not enough for a person to leave when he "holds" one of his most preferred objects. With the description of our algorithm, we state a more sophisticated condition that needs to be met for a person (or group of people) to be removed. This aspect of our algorithm is crucial to achieving a *Pareto-efficient* allocation.

   We address this by defining the following condition for departure:: A group of people can leave only if there is no trade with people outside of the group that can make someone outside the group better off without hurting someone inside the group. This condition ensures *Pareto-efficiency* of the final allocation. In previous example, if 2 gets a, he can leave with it since any further trade would make him worse off. However, 1 cannot leave with b: there is a trade, between 1 and 3 that would make 3 better off without hurting 1. Thus, after 1 and 3 trade, 1 leaves with c and 3 with b.

2. When a person is indifferent between objects, where does he point? This is a more difficult question. We explain how to deal with this issue in a way that does not compromise *strategy-proofness*. In addition, our condition guarantees that the algorithm terminates.

   Our answer is to define the following condition for pointing: A natural way to solve the problem is to use a priority order over the people. However, naïvely breaking ties according to a fixed priority order does not work with our condition for departure. We illustrate this through the following examples.

[5]Regardless of how we break ties, the result of the *top-trading cycles* algorithm is a "competitive allocation" [3]. In fact every *competitive allocation* can be found in this way. However, as evidenced by the above example, for some profiles of preferences, no *competitive allocation* is *Pareto-efficient*.

[6]We have the "departure phase" at the beginning of each step rather than at the end. We have done this for expositional simplicity since it rules out people "pointing" at themselves.

(a) Let $N = \{1, 2, 3\}$, $\omega = (a, b, c)$, and $R \in \mathcal{R}^N$ be as follows.

R_1	R_2	R_3
$a\ b\ c$	$a\ b$	c
	c	$a\ b$

Let $1 \prec 2 \prec 3$. Suppose each person cannot point at himself when he is indifferent between what he holds and what someone else holds.[7] In this case, 1 and 2 trade at each step between them and the condition of departure is never satisfied. The algorithm does not terminate.

To guarantee that the algorithm terminates, the priority order defined in the algorithm is updated at every stage to give higher priority to people who do not hold one of their most preferred objects than to people who hold one of their most preferred objects.

(b) Let $N = \{1, 2, 3, 4, 5\}$, $\omega = (a, b, c, d, e)$ and $R \in \mathcal{R}^N$ be as follows.

R_1	R_2	R_3	R_4	R_5
$c\ d\ e$	a	a	e	$b\ c$
a	e	c	d	e
	b			

Let $1 \prec 2 \prec 3 \prec 4 \prec 5$. If we use a naïve pointing scheme, in the first step, 1 points at 3 and 5 points at 2. Then, 1 and 3 trade and 3 leaves with a. In the second step, since $1 \prec 2$, 5 points at 1 and since $4 \prec 5$, 1 points at 4. Then, 1, 4, and 5 trade. The algorithm terminates and 2 is left with b. However, if 2 reports $\begin{array}{c} R'_2 \\ \hline e \\ b \end{array}$, 2's assignment is e. Therefore, 2 is better off reporting the lie R'_2 when his true preference relation is R_2 and others' preferences are R_{-2}.

To guarantee that the rule is *strategy-proof*, at each step, each i points at the same person that he pointed at in the previous step as long as that person holds the same object (that is, he did not trade).

We bring these ideas together and define a modification of the *top-trading cycles* algorithm.

For each $(R, \omega) \in \mathcal{R}^N \times A$, we define the allocation selected by **top cycles rule with priority $\prec$, $TC^{\prec}(R, \omega)$,** via the following algorithm.

[7] Otherwise, 1 will point to himself at each step of the algorithm. Since the departure condition is not met, the algorithm does not terminate.

Each step of the algorithm proceeds in three phases: departure, pointing, and trading. The goal of the algorithm is to enlarge, at each step, the set of "satisfied" people: those holding one of their most preferred objects. However, we aim to do this in a way that provides incentives for every person to report his true preferences. To achieve this, the algorithm favors "unsatisfied" people, those who are not satisfied, who have higher priority by connecting more people to them (via direct or indirect pointing) as compared to people with lower priority.

In the first step, the set of remaining people is N and each $i \in N$ holds $\omega(i)$.

1. **Departure:** A group of people is chosen to "depart" if two conditions are met.

   i) What each person in the group holds is among his most preferred objects (among the remaining ones), and

   ii) All of the most preferred objects (among the remaining ones) of the group are held by them.

   Once a group departs, each of them is assigned one of his top objects and is removed from the set of *remaining people*. In addition, their objects are removed from the *remaining objects*. There may be another group that can be chosen to depart. The process continues until there are no more groups that can depart. If the two conditions are not met by any group, then nobody departs.

2. **Pointing:** Each person points at a person holding one of his top objects (among the remaining ones). Since there may be more than one such person, the problem of figuring whom each person points at is a complicated one. We solve it in stages as follows:

Stage 1) For each remaining j who holds the same object that he held in the previous step, each i that pointed at j in the previous step points at j in the current step. Of course, this does not apply for the very first step.

Stage 2) Each i with a unique top object (among the remaining ones) points at the person holding it.

Stage 3) Each person who has at least one of his top objects (among the remaining ones) held by an *unsatisfied* person points at whomever has the highest priority among such *unsatisfied* people.

Stage 4) For each person who is not yet pointing at anyone, his most preferred objects are all held by *satisfied* people. If at least one of his top objects (among the remaining ones) held by a *satisfied* person who points at an *unsatisfied* person points at whomever points at the *unsatisfied* person with highest priority. If two or more of his *satisfied* "candidates" point at the *unsatisfied* person with highest priority, he points at the *satisfied* candidate with the highest priority.

Stage $\cdots$) And so on.

3. **Trading:** Since each *remaining* person points at someone, there is at least one cycle of remaining people. For each such cycle, people trade according to the way that they point and what they hold for the next step is updated accordingly.

The algorithm terminates when everyone has departed.

To see that the algorithm terminates, note that at each step, since N is finite and there is at least one cycle involving an unsatisfied person, either

1. At least one person departs with his holding, or
2. At least one person's holding is switched to an object that he ranks highest among those remaining. That is, at least one person becomes *satisfied*.

Therefore, the algorithm terminates in a finite number of steps.[8] Since the algorithm terminates and provides a unique allocation for every problem, $TC^{\prec}$ is a well-defined rule.

A rigorous formal description of the algorithm is in Appendix A.

To help illustrate the *top cycles rule*, we provide Example 2. First, we state some useful definitions.[9] In the t^{th} step, after the *departure phase*, there is a set of **remaining objects,** $\boldsymbol{O_t \subseteq O}$, and **remaining people,** $\boldsymbol{N_t \subseteq N}$. Each *remaining* person, $i \in N_t$, **holds** the object $\boldsymbol{h_t(i)}$.[10] For each $i \in N_t$, the **person**

[8]The algorithm runs in polynomial time. Specifically, it runs in $O(|N|^5)$ where $|N|$ is the number of people in the problem: At each step, at least one person becomes *satisfied* or departs. Thus, the algorithm runs for $O(|N|)$ steps. Each step consists of a departure phase, a pointing phase, and a trading phase. The largest group that can depart can be found by starting with the set of *satisfied* people. The departure condition is checked (this is an $O(|N|^2)$). We stop if it is met. Otherwise, we drop a person who has one of his most preferred objects held by someone outside the group ($O(|N|^2)$). We do this either until we find a group that can depart or we have dropped everyone. Thus the condition is checked and a person to drop is sought $O(|N|)$ times. After the largest group is found, another group is sought. This is repeated $O(|N|)$ times. Thus, the departure phase runs in $O(|N|^4)$. The pointing phase has $O(|N|)$ stages, each of which runs in $O(|N|^2)$. The trading phase runs in $O(|N|)$.

[9]These, along with notation introduced later in the paper, are summarized in Appendix D.

[10]Note that for each i, $h_t(i)$ denotes the object that i holds at the beginning of Step t, while $h_{t+1}(i)$ denotes the object i is holding at the end of Step t. If i trades in Step t, the object that he holds in Step t, $h_t(i)$, is different from the one he has at the end of Step t and beginning of Step $t+1$, $h_{t+1}(i)$. If i does not trade $h_t(i) = h_{t+1}(i)$.

whom i points at, $p_t(i)$. For notational convenience, for each $i, j \in N_t$, we use $i \underset{t}{\longrightarrow} j$ to denote $p_t(i) = j$. If $p_t(p_t(i)) = j$, we write $i \underset{t}{\longrightarrow}\underset{t}{\longrightarrow} j$, and so on. Given $M \subseteq N_t$, if $p_t(i) \in M$, we write $i \underset{t}{\longrightarrow} M$. If $p_t(p_t(i)) \in M$, we write $i \underset{t}{\longrightarrow}\underset{t}{\longrightarrow} M$, and so on.

At each step, the set of remaining people is partitioned into two sets: **satisfied people, S_t** $\equiv \{i \in N_t : h_t(i) \in \tau(R_i, O_t)\}$, who hold an object that they rank at the top of the remaining objects, and **unsatisfied people** who do not, $U_t \equiv N_t \setminus S_t$.

Example 2. *Top cycles rule with fixed tie breaking.*

Let $O = \{a, b, c, d, e, f, g, h, i, j, k\}$, and $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. Consider $(R, \omega) \in \mathcal{R}^N \times A$ such that $\omega = (a, b, c, d, e, f, g, h, i, j, k)$ and $R \in \mathcal{R}^N$ as follows:

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}
a	a	f	$c\ d\ e\ f$	$d\ e\ g$	$d\ e$	d	$d\ h\ i$	$c\ i$	$a\ b\ j$	$e\ i\ k$
$\vdots$	b	$\vdots$	$\vdots$	$\vdots$	$\vdots$	g	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$\vdots$					$\vdots$				

Let $\prec$ be such that $1 \prec 2 \prec 3 \prec 4 \prec 8 \prec 5 \prec 6 \prec 7 \prec 9 \prec 10 \prec 11$.

We start with $O_0 = O, N_0 = N$, and $h_1 = \omega$.

Step 1:

Departure phase: The first group to depart is $\{1\}$. To see this, note that 1's most preferred object in O is the unique object a, his endowment. Given that 1 leaves with a, the second group to departs is $\{2, 10\}$, 2's most preferred object in $O \setminus \{a\}$ is the unique object b and 10's most preferred objects in $O \setminus \{a\}$ are b and j. Now, $TC^{\prec}(R, \omega)(1) = a$, $TC^{\prec}(R, \omega)(2) = b$, and $TC^{\prec}(R, \omega)(10) = j$. Further, the *remaining people* are $N_1 = \{3, 4, 5, 6, 7, 8, 9, 11\}$, and the *remaining objects* are $O_1 = \omega(N_1)$. From this, the *satisfied people* are $S_1 = \{4, 5, 8, 9, 11\}$ and the *unsatisfied people* are $U_1 = \{3, 6, 7\}$.

Pointing phase: This is illustrated in Figure 2.

Stage 1) Not applicable to the first step.

Stage 2) Each person with a unique most preferred object in O_1 points at whomever holds that object. In this case, $3 \underset{1}{\longrightarrow} 6$ and $7 \underset{1}{\longrightarrow} 4$.

Stage 3) Each person such that at least one of their most preferred objects is held by an unsatisfied person points at an unsatisfied person. Such people are 4, 5, and 9. Since 5 and 9 have only one unsatisfied person to point at, they point accordingly. That is, $5 \underset{1}{\longrightarrow} 7$ and $9 \underset{1}{\longrightarrow} 3$. However, 4 is indifferent between the objects held by 3 and 6. In accordance with $\prec$, $4 \underset{1}{\longrightarrow} 3$.

Stage 4) Each person whose most preferred objects are held by satisfied people pointing at an unsatisfied person. 6, 8, and 11 are such people. Since 4 and 5 hold 6's most preferred objects, we consider who they are pointing at. Since $4 \underset{1}{\longrightarrow} 3, 5 \underset{1}{\longrightarrow} 7$, and $3 \prec 7$, we have $6 \underset{1}{\longrightarrow} 4$ rather than $6 \underset{1}{\longrightarrow} 5$. 8's preferred objects (among the remaining ones) are hold by 9 and 4. Note that both point at 3. Since $4 \prec 9$, $8 \underset{1}{\longrightarrow} 4$. Since 5 and 9 hold 11's most preferred objects, $5 \underset{1}{\longrightarrow} 7, 9 \underset{1}{\longrightarrow} 3$, and $3 \prec 7$, we have $11 \underset{1}{\longrightarrow} 9$

Trading phase: We observe that there is only one cycle and it involves 3, 4, and 6. Thus, $h_2 = (-, -, f, c, e, d, g, h, i, -, k)$. This trade results in, 3 and 6 becoming *satisfied* in the next step.

Step 2:

Departure phase: The only group satisfying the departure condition is $\{3\}$. From this, $TC^{\prec}(R, \omega)(3) = f$, $N_2 = \{4, 5, 6, 7, 8, 9, 11\}$, $O_2 = \{c, d, e, g, h, i, k\}$, $S_2 = \{4, 5, 6, 8, 9, 11\}$, and $U_2 = \{7\}$..

Pointing phase: This is illustrated in Figure 2.

Stage 1) Note that 5 was pointing at 7 in Step 1 and 7 is holding the same object. That is, $5 \underset{1}{\longrightarrow} 7 \in N_2$ and $h_2(7) = h_1(7)$. Then, we have $5 \underset{2}{\longrightarrow} 7$. In addition, since $11 \underset{1}{\longrightarrow} 9 \in N_2$ and $h_2(9) = h_1(9)$, we have $11 \underset{2}{\longrightarrow} 9$.[11]

Stage 2) Since 7's unique most preferred object is d, $7 \underset{2}{\longrightarrow} 6$.

Stage 3) No person, other than 5, most prefers g (7's holding) among O_2.

Stage 4) 4 and 6 point at 5 whose pointing to an unsatisfied person: $4 \underset{2}{\longrightarrow} 5$ and $6 \underset{2}{\longrightarrow} 5$.

Stage 5) 8 and 9 are pointing to someone that is pointing to 5. Thus, $8 \underset{2}{\longrightarrow} 6$ and $9 \underset{2}{\longrightarrow} 4$.

Trading Phase: At the end of this Step, there is one cycle and it involves 5, 6, and 7. In the *trading phase*, we get $h_3 = (-, -, -, c, g, e, d, h, i, -, k)$.

[11] Without Stage 1, 11 would point to 5 who is pointing at an unsatisfied person.

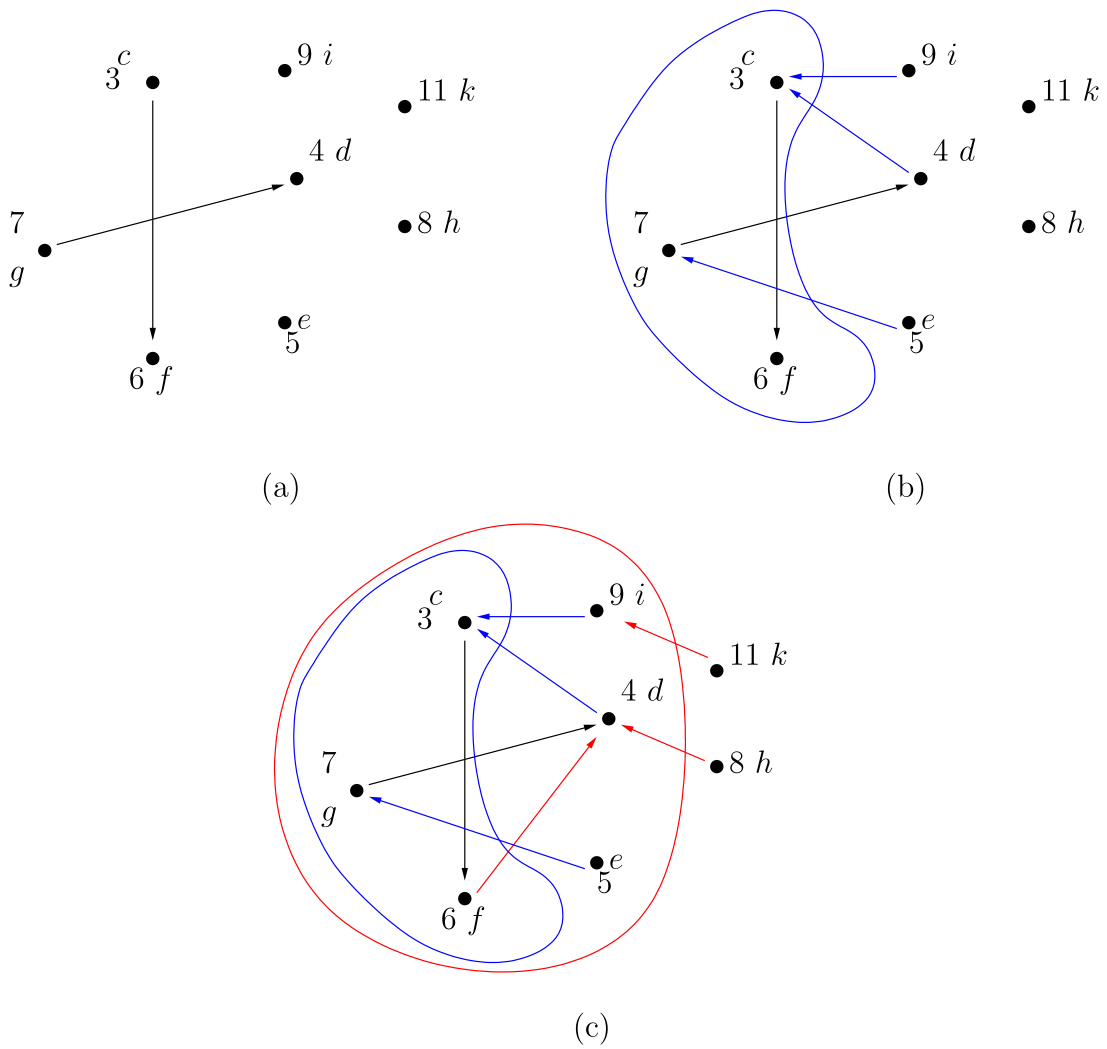


Figure 1: Pointing phase of Step 1: (a) Since 3 and 7 have unique most preferred objects, they point at whoever holds those objects. (b) Next, we consider 4, 9 and 5: those who have a most preferred object that is held by an unsatisfied person in the bubble. (c) Finally, we consider 6, 8, and 11: those who have a most preferred object that is held by a member of the bigger bubble: people who can point at an unsatisfied person.

Step 3:

Departure phase: We end after the departure phase of Step 3 since all the remaining people, $\{4, 5, 6, 7, 8, 9, 11\}$, satisfy the conditions. Then, $N_3 = \varnothing$ y $O_3 = \varnothing$.

Thus, $TC^{\prec}(R, \omega) = (a, b, f, c, g, e, d, h, i, j, k)$. •

In the next section, we show that $TC^{\prec}$ is *strategy-proof*, *Pareto-efficient*, and *individually rational*. We also show that $TC^{\prec}$ always picks an allocation from the *weak core*.

When the input preference profile does not involve any indifference, the priority order $\prec$ plays no role in the definition of $TC^{\prec}$ since for each $i \in N$ and each t, $\tau(R_i, O_t)$ is a singleton and $p_t(i)$ is defined in the first two stages of the *pointing phase*. Thus, for each $(P, \omega) \in \mathcal{P}^N \times A$ and each pair of priority orders $\prec$ and $\prec'$, $TC^{\prec}(P, \omega) = TC^{\prec'}(P, \omega)$.

Remark 1. It is natural to ask whether, for each $(R, \omega) \in \mathcal{R}^N \times A$, there is a corresponding problem $(P', \omega) \in \mathcal{P}^N \times A$ such that,

1. For each $i \in N$ and each pair $x, y \in O$, if $x \; P'_i \; y$, then $x \; R_i \; y$, and
2. $TC^{\prec}(R, \omega) = TC^{\prec}(P', \omega)$.

However, this is not the case. Let us go back to Example 1. For each $\prec$ such that $2 \prec 3$, $TC^{\prec}(R, \omega) = (c, a, b)$, and for each $\prec'$ such that $3 \prec' 2$, $TC^{\prec'}(R, \omega) = (b, c, a)$. But $TC^{\prec}(P^1, \omega) = (b, a, c)$ and $TC^{\prec}(P^2, \omega) = (c, b, a)$, neither of which coincides with $TC^{\prec}(R, \omega)$ or $TC^{\prec'}(R, \omega)$. [12] ○

6 Results

We first observe that *strategy-proofness* and *Pareto-efficiency* are incompatible with *anonymity*. We also note that the additional requirement of *individual rationality* leads to an incompatibility with *non-bossiness*. We then show that these incompatibilities are tight by proving that *top cycles rules with fixed tie breaking* satisfy all three of our central axioms. We omit the proofs of these results since they follow from those of Bogomolnaia, Deb, and Ehlers [5].

Proposition 1. *If $N > 2$, no rule is strategy-proof, Pareto-efficient and anonymous.*

[12] As evidenced by the above example, $TC^{\prec}(R, \omega)$ need not to be a competitive allocation.

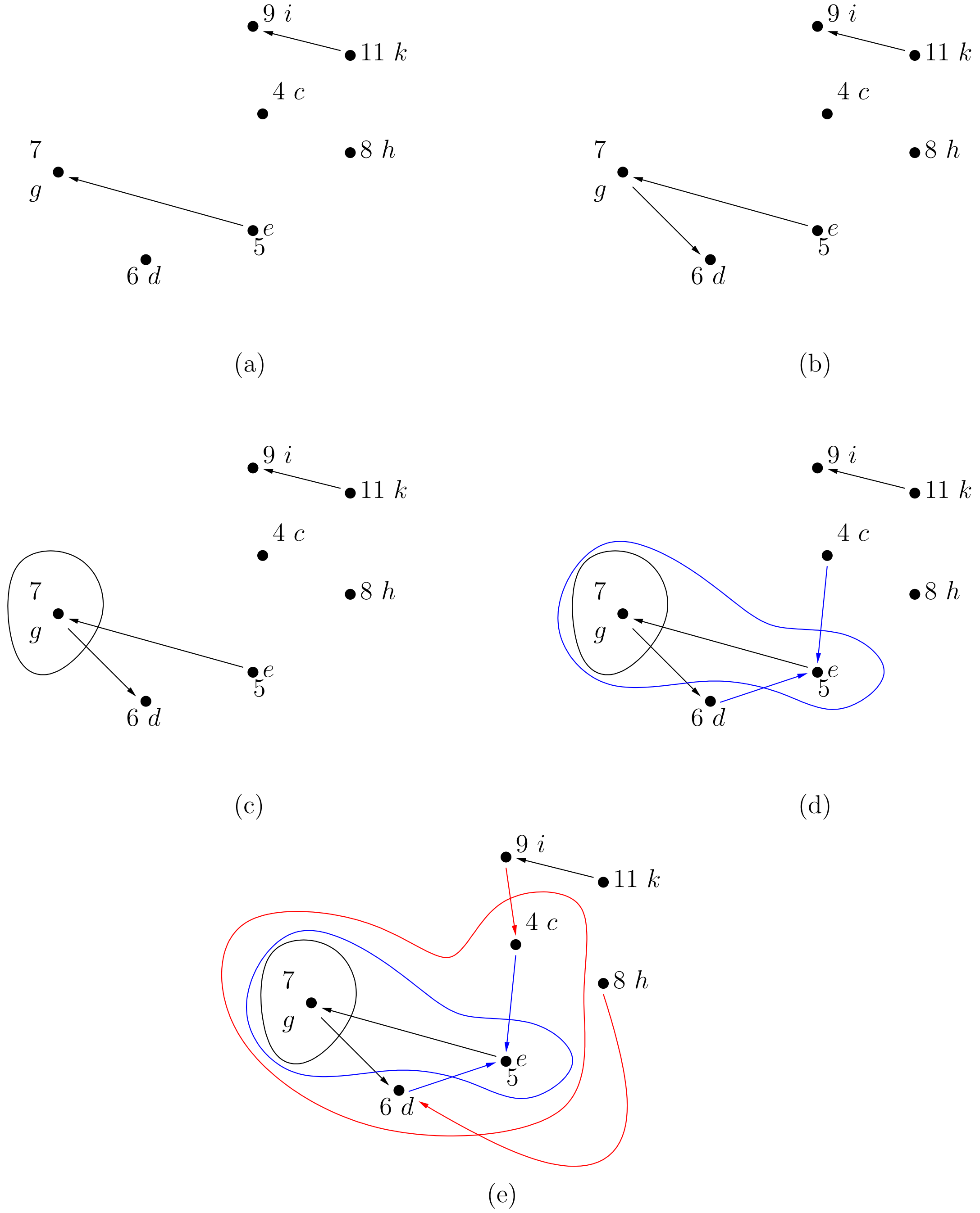


Figure 2: Pointing phase of Step 2: (a) Since 5 pointed at 7 in Step 1 and 7 has not traded, 5 points at 7 in Step 2 as well. Similarly, 11 points at 9 in Step 2 as well. Without Stage 1, 11 would point to 5 in Step 2. (b) 7 is the only person with a unique most preferred object. (c) We now consider people who can point at the only unsatisfied person, 7. However, there is no such person. (d) Next, we consider 4 and 6 who point into the bubble containing 7 and 5. (e) Finally, 8 and 9 point into the biggest bubble.

Proposition 2. *If $N > 2$, no rule is strategy-proof, Pareto-efficient, individually rational, and non-bossy.*[13]

Next, we show that both Propositions 2 and 1 are tight. When we drop *non-bossiness* or *anonymity* from the lists of requirements, the incompatibility does not persist, as evidenced by *top cycles rules with fixed tie breaking*.

By definition *top cycles rules with fixed tie breaking* are not anonymous. To see that they are *bossy*, consider the following example.

Example 3. Bossiness of top cycles rules with fixed tie breaking: Let $O = \{a, b, c\}, N = \{1, 2, 3\}, \omega = (a, b, c)$, and $1 \prec 2 \prec 3$. Let $R, R' \in \mathcal{R}^N$ be such that,

R_1	R_2	R_3
$a\ b\ \textcircled{c}$	$\textcircled{a}$	a
	b	$\textcircled{b}$
	c	c

R'_1	R_2	R_3
$\textcircled{c}$	a	$\textcircled{a}$
$a\ b$	$\textcircled{b}$	b
	c	c

.

Then, $TC^{\prec}$ selects the circled allocations above, showing that it is *bossy*. •

Proposition 3. *For each priority order $\prec$, $TC^{\prec}$ is Pareto-efficient and individually rational. That is, for each $(R, \omega) \in \mathcal{R}^N \times A$ and each $\prec$, $TC^{\prec}(R, \omega) \in PE(R) \cap IR(R, \omega)$.*

Proof: By definition of $TC^{\prec}$, it is *individually rational*.

We show that it is *Pareto-efficient* using the conditions of the Departure Phase in the algorithm. Consider the sequence of groups of people who leave at the first step. By condition (i) of the Departure Phase, each member of the first group leaves with one of his most preferred objects. By condition (ii) of the Departure Phase, each of them can be made no better off. By the same reasoning, each member of the second group leaves with one of his most preferred objects after members of the first group have left and can be made no better off without hurting at least one member of the first group. Continuing, each member of a group that in Step 1 leaves with one of his most preferred objects after members of all the previous groups have left and can be made no better off without hurting at least one person who has left.

A similar argument applies to the subsequent steps. Those leaving in later steps can be made no better off without hurting those who have left in prior steps. Thus, $TC^{\prec}$ is *Pareto-efficient*. ■

Proposition 4. *For each priority order $\prec$, $TC^{\prec}$ selects an allocation from the weak core. That is, for each $(R, \omega) \in \mathcal{R}^N \times A$ and each $\prec$, $TC^{\prec}(R, \omega) \in C^W(R, \omega)$.*

[13]This is a corollary of Theorem 2 in [5]. We provide a direct proof upon request.

Proof: Suppose not. Then, there are $(R,\omega) \in \mathcal{R}^N \times A$ and $S \subseteq N$ *such that* S *blocks* $\alpha \equiv TC^{\prec}(R,\omega)$. That is, there is $\beta \in A$ such that $\beta(S) = \omega(S)$ and for each $i \in S$, $\beta(i)\ P_i\ \alpha(i)$.

For each t and each $i \in S$, if $\beta(i) \in O_t$, then there is no $j \in N$ such that i points at him in Step t ($i \underset{t}{\not\longrightarrow} j$) and $\beta(i)\ P_i\ h_t(j)$.

Let $\hat{t}$ be the first step at which there is $i \in S$ such that i is part of a trading cycle at the end of step $\hat{t}$. Once i trades, his welfare is determined and he is made neither better nor worse off during the remainder of the algorithm. Thus, he is indifferent between the object that makes his first trade for and the object he ends up with. That is, $h_{\hat{t}+1}(i)\ I_i\ \alpha(i)$. So $\beta(i)\ P_i\ h_{\hat{t}+1}(i)$. This implies that there is $j \in N$ such that i points at j in Step $\hat{t}$ ($i \underset{\hat{t}}{\longrightarrow} j$) and $\beta(i)\ P_i\ h_{\hat{t}}(j) = h_{\hat{t}+1}(i)$. Thus, $\beta(i) \notin O_{\hat{t}}$. However, since $\beta(S) = \omega(S)$, there is $k \in S$ such that $\beta(i) = \omega(k)$ and since $\omega(k) \notin O_{\hat{t}}$, k is part of a trading cycle at some $\tilde{t} < \hat{t}$. This contradicts the definition of $\hat{t}$. ■

In order to show that for each $\prec$, $TC^{\prec}$ is *strategy-proof*, we make a preliminary remark and state two key lemmas.

For each problem $(R,\omega) \in \mathcal{R}^N \times A$, the "state" of the algorithm at Step t is summarized by the tuple (O_t, N_t, h_{t+1}, p_t). Our remark and lemmas pertain to how these tuples change in response to changes in the input problem. These lemmas provide useful insight into the dynamics of the algorithm.

Remark 2. (Persistence) *If i points at j at Step t, then he points at j as long as j holds the same object. That is, if $i \underset{t}{\longrightarrow} j$, then for every $t' > t$ such that $h_{t'}(j) = h_{t'-1}(j) = h_{t'-2}(j) = \ldots = h_t(j)$, $i \underset{t'}{\longrightarrow} j$.*

Before we proceed to our first lemma, we introduce some additional notation. Let $(R,\omega) \in \mathcal{R}^N \times A$ and $i \in N$. At the Step t of the algorithm, let the set of people **connected to $\boldsymbol{i}$, $\boldsymbol{CONN(i,R,t)}$**, be those, including i, connected to i via p_t. That is,

$$CONN(i,R,t) = \left\{ j \in N_t : \begin{array}{l} j \equiv i, \text{ or} \\ j \underset{t}{\longrightarrow} i, \text{ or} \\ j \underset{t}{\longrightarrow}\underset{t}{\longrightarrow} i, \text{ or} \\ \ldots \end{array} \right\}.$$

Fix $\omega \in A$ and priority order $\prec$. Let $R \in \mathcal{R}^N$, $i \in N$, and $R'_i \in \mathcal{R}$. Let $R' = (R'_i, R_{-i})$. For each $\hat{t} = 0, 1, \ldots$, let $h_{\hat{t}}$ be the *holding vector* at Step $\hat{t}$ of the algorithm for the problem (R,ω). Similarly define $h'_{\hat{t}}$ for the problem (R',ω). We also define, $O_{\hat{t}}, O'_{\hat{t}}$, $N_{\hat{t}}, N'_{\hat{t}}, p_{\hat{t}}, p'_{\hat{t}}, S_{\hat{t}}, S'_{\hat{t}}$, $U_{\hat{t}}$, and $U'_{\hat{t}}$. Finally, for each $i,j \in N$, we indicate $p_{\hat{t}}(i) = j$ by $i \xrightarrow[\hat{t}]{R} j$ and $p'_{\hat{t}}(i) = j$ by $i \xrightarrow[\hat{t}]{R'} j$. We also use $i \not\xrightarrow[\hat{t}]{R} j$ to indicate $p_{\hat{t}}(i) \neq j$.

Let t be the step at which i either leaves or makes his first trade under R. Define t' similarly with respect to R'. Let $\bar{t}$ be the first step at which i is satisfied for exactly one of the two problems if such a number exists, and ∞ otherwise. That is, $i \in S_{\bar{t}}$ and $i \in U'_{\bar{t}}$, $i \in U_{\bar{t}}$ and $i \in S'_{\bar{t}}$, or $\bar{t} = \infty$.[14]

Let $\underline{t} \equiv \min\{t, t', \bar{t}\}$. Then, $\underline{t}$ determines the first period when i is satisfied under R or R'.

Our first lemma states that up to Step $\underline{t}$, there is no difference in the *state* of the algorithm, regardless of whether i reports R_i or R'_i.

Lemma 1. **($\underline{t}$ equality)** *At $\underline{t}$, for both R and R', the objects and people remaining, as well as the holding vector and previous step's pointing vector, except for i's component, are the same. That is,*

$$\begin{matrix} O_{\underline{t}} & N_{\underline{t}} & h_{\underline{t}} \\ \shortparallel & \shortparallel & \shortparallel \\ O'_{\underline{t}} & N'_{\underline{t}} & h'_{\underline{t}} \end{matrix} \quad \textit{and for each } j, k \in N_{\underline{t}} \textit{ such that } j \neq i, \quad \begin{matrix} j \xrightarrow[\underline{t}-1]{R} k \\ \Updownarrow \\ j \xrightarrow[\underline{t}-1]{R'} k. \end{matrix}$$

While the formal proof is Appendix B, Example 4 should help the reader build some intuition with regards to Lemma 1.

Example 4. *Lemma $\underline{t}$-equality.*

Let $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $\omega = (a, b, c, d, e, f, g, h)$, and $R \in \mathcal{R}^N$ be as follows.

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
$c\ e$	$a\ b\ c$	e	$b\ d$	$b\ e\ f$	e	$d\ f\ g$	$c\ h$
f		h					

Let $\prec$ be such that $1 \prec 2 \prec 3 \prec 4 \prec 5 \prec 7 \prec 8$. Figure 3. shows the pointing stages of Step 1, 2, and 3. Notice that 1 does not trade until Step 3. Suppose that, instead of R_1, 1 reports R'_1 such that he is unsatisfied. We will show that the departure, pointing, and trading stages remain unaffected until 1 trades under one of the two announcements.

Since 1 is unsatisfied, no one leaves in the departure phase of Step 1 under $R'(\equiv (R'_1, R_{-1}))$. Moreover, each person who points at 1 under R points at him under R'. That is, only 2 points at 1. All but 1 point the same under R and R'. Therefore, all the trading cycles not involving 1 that are realized under R are

[14]The last case, $\bar{t} = \infty$, only occurs if $t = t'$.

also realized under R'. If 1 is not part of a cycle (then he does not trade) under R', then the objects, people, and holdings are the same under R and R' at the beginning of Step 2.

If 1 is unsatisfied under R' in Step 2, the departure stage of Step 2 is the same under R and R'. Since 1 is unsatisfied under R and R', people who point at 1 do not change. That is, 2 still points at 1. All but 1 again point in the same way. All the trading cycles not involving 1 are realized. If 1 does not trade under R' then the objects, people, and holdings are the same under R and R' at the beginning of Step 3.

Following the same argument as in the previous steps, we conclude that under R and R' the departure phase is the same. And until 1 trades, the objects, people, and holdings are the same at the beginning of the the next Step. Note that the $\underline{t}$-equality Lemma does not say anything about the pointing phase and trading phase of Step 3 when 1 trades. •

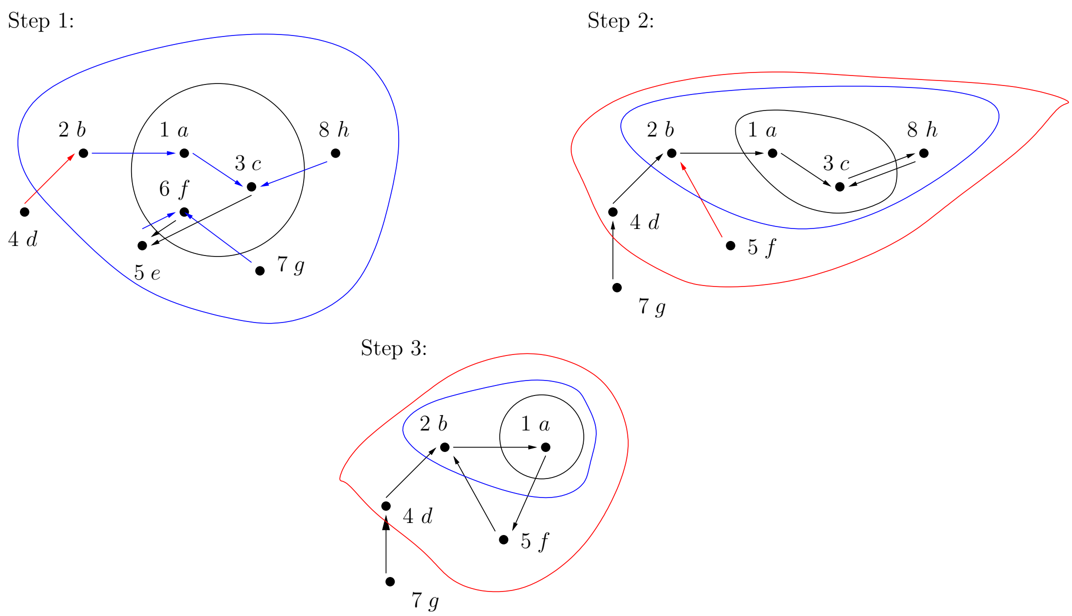


Figure 3: Example of $\underline{t}$-equality Lemma. If 1 reports R'_1 rather than R_1, then each stage in Step 1 and Step 2 remain the same.

For each $R_0 \in \mathcal{R}$, and each $a \in O$, let the **indifference class of $\boldsymbol{a}$ at $\boldsymbol{R_0}$, $\boldsymbol{I(a, R_0)}$**, be

$$I(a, R_0) = \{b \in O \mid b \; I_0 \; a\}.$$

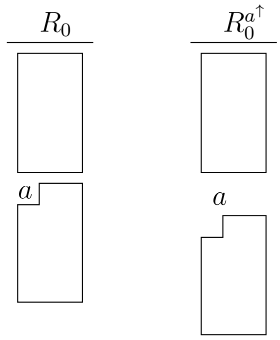


Figure 4: *Local push-up* of a preference relation: Given $R_0 \in \mathcal{R}$ and $a \in O$, the *local push-up of R_0 at a, $R_0^{a\uparrow}$* is as shown above.

Given $R_0 \in \mathcal{R}$, and $a \in O$, let the **local push-up of $\boldsymbol{R_0}$ at $\boldsymbol{a}$, $\boldsymbol{R_0^{a\uparrow}}$** $\in \mathcal{R}$ be the relation that differs from R_0 *only* in that it ranks a above all objects in $I(a, R_0)$, as shown in Figure 3. That is,

$$R_0|_{O\setminus\{a\}} = R_0^{a\uparrow}|_{O\setminus\{a\}} \text{ and for each } b \in O \setminus \{a\}, \begin{array}{l} b \; P_0 \; a \Rightarrow b \; P_0^{a\uparrow} \; a \text{ and} \\ a \; R_0 \; b \Rightarrow a \; P_0^{a\uparrow} \; b. \end{array}$$

To prove that $TC^{\prec}$ is *strategy-proof* we will have to consider all possible preference relations that a person can misreport. However, we can split all the available misreports into two categories. The first category includes only preference relations under which the person is not indifferent between the object that he is assigned and any other object. The second category consists of all the remaining preference relations. The following lemma implies that for each preference relation in the second category, we can find a preference relation in the first category such that the person is assigned the same object regardless of which of the two preference relations he reports. We use this to prove that $TC^{\prec}$ is *strategy-proof* since it means that we only need to rule out successful misreports from the first category.

Consider the case in which i's preference relation changes from R_i to R'_i. Moreover, R'_i is a *local push-up* of R_i at a. Note that if $\bar{t} < \min\{t, t'\}$, i becomes satisfied in only one of the two problems. Since the only difference between these preference relations is that $I(a, R'_i) \subset I(a, R_i)$, then, by the $\underline{t}$ *equality* lemma, $\tau_i(R', O'_{\bar{t}}) \subset \tau_i(R, O_{\bar{t}})$. Moreover, $i \in S_{\bar{t}}$, $i \in U'_{\bar{t}}$, and $a \in \tau_i(R, O_{\bar{t}}) \cap \tau_i(R', O'_{\bar{t}})$. We use this fact to prove the following lemma. In the proof, we follow the same structure as in the proof of the $\underline{t}$ *equality* lemma. That is, we establish the state of the algorithm between $\bar{t}$ and $\min\{t, t'\}$, and then between t and t'.

Lemma 2. (Invariance) *If the preference relation of a person changes to a local push-up of his original preference at his assignment, then his assignment is* *unchanged. That is, if* $\alpha = TC^{\prec}(R,\omega)$, $R'_i = R_i^{\alpha(i)\uparrow}$, *and* $\alpha' = TC^{\prec}(R',\omega)$, *then* $\alpha(i) = \alpha'(i)$.

Proof: By the $\underline{t}$ *equality* lemma and by the definition of R_i and R'_i, in each step before $\underline{t}$, i points to the same person (holding the same object) under R or R'. Thus, $O_{\underline{t}} = O'_{\underline{t}}$, $N_{\underline{t}} = N'_{\underline{t}}$, $h_{\underline{t}} = h'_{\underline{t}}$, and for each $j \in N_{\underline{t}} \setminus \{i\}$, $p_{\underline{t}-1}(j) = p'_{\underline{t}-1}(j)$. Since R'_i is a *local push-up* of R_i at α_i, $\tau(R'_i, O_{\underline{t}}) = \{\alpha(i)\} \subseteq \tau(R_i, O_{\underline{t}})$.
The $\underline{t}$ *equality* lemma also implies that $CONN(i,R,\underline{t}-1) = CONN(i,R',\underline{t}-1)$.

The rest of the proof proceeds as follows. First we show that at $\min\{t,t'\}$, any person connected to i under R is connected to i under R'. Then, we show that i's component of the allocation chosen under R is the same as his component of the allocation chosen under allocation under R': $\alpha'(i) = \alpha(i)$.

Claim 1. ***(Pre-trade inclusion)***[15] *For each* $\ddot{t} = \underline{t}, ..., \min\{t,t'\}$,[16]

(i) The objects and people remaining at $\ddot{t}$ *under* R *are a subset of those remaining under* R'*. Further, those remaining under* R' *but not under* R *are connected to* i*. That is,*

$$O_{\ddot{t}} \subseteq O'_{\ddot{t}}, \qquad N_{\ddot{t}} \subseteq N'_{\ddot{t}}$$
$$O'_{\ddot{t}} \setminus O_{\ddot{t}} \subseteq h_{\ddot{t}}(CONN(i,R',\ddot{t}-1)), \text{ and } N'_{\ddot{t}} \setminus N_{\ddot{t}} \subseteq CONN(i,R',\ddot{t}-1).$$

(ii) Every person who is satisfied at $\ddot{t}$ *under* R' *is satisfied under* R*. Every person who is not satisfied under* R' *but is satisfied under* R *is connected to* i *under* R'*. That is,* $S'_{\ddot{t}} \subseteq S_{\ddot{t}}$ *and* $S'_{\ddot{t}} \setminus S_{\ddot{t}} \subseteq CONN(i,R',\ddot{t}-1)$.

(iii) Every person not connected to i *at* $\ddot{t}$ *under* R' *points at the same person under* R *as under* R'*. That is, for each* $j \in N'_{\ddot{t}} \setminus CONN(i,R',\ddot{t})$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$.

(iv) Every person not connected to i *at* $\ddot{t}$ *under* R' *holds the same object under* R *as under* R'*. That is, for each* $j \in N'_{\ddot{t}} \setminus CONN(i,R',\ddot{t})$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$.

(v) The set of people connected to i *under* R *is a subset of the people connected to* i *under* R'*. That is,* $CONN(i,R,\ddot{t}) \subseteq CONN(i,R',\ddot{t})$.

While the formal proof is in Appendix C, Example 5 should help the reader build some intuition with regards to Claim 1.

Example 5. *Pre-trade Claim.*
Let $N = \{1,2,3,4,5,6,7,8\}$, $\omega = (a,b,c,d,e,f,g,h)$, and $R \in \mathcal{R}^N$ be as follows.

[15] As illustrated in Figure 5.
[16] If $\underline{t} = \min\{t,t'\}$ statements (i) - (v) are implied by the $\underline{t}$ *equality* lemma.

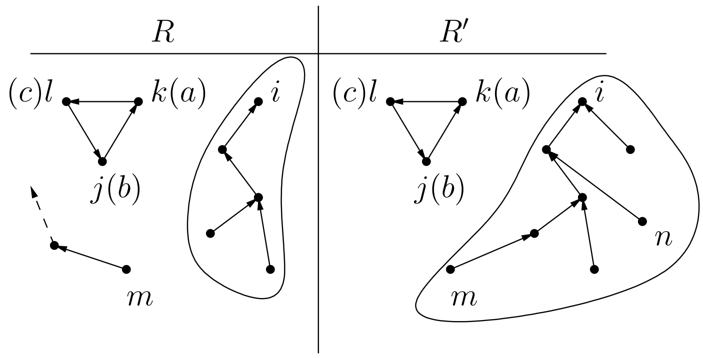


Figure 5: Pre-trade inclusion.

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R'_1
a b c	c	e	e	a d e	a f	h	e g h	b
		d						a c

The Pre-trade Claim deals with situations where 1 reports, instead of R_1, a *local push-up* of R_1 at his assignment $TC^{\prec}(R,\omega)(1) = b$, R'_1. The Pre-trade Claim says that at each step before 1 trades under $R'(\equiv (R'_1, R_{-1}))$, the set of people who point at 1 under R is a subset of the set of people who point at 1 under R'. Moreover, trades that do not involve 1 that occur under R occur under R' as well.

In this example, the departure stage of Step 1 is the same under R and R' (figure 6.) In the pointing stage, since 1 is satisfied under R but not under R', more people point at 1 under R' than under R. Under R', 5 points at 1. In addition, note that since 6 points at 1 under R, where 1 is satisfied, he also points at 1 under R', where 1 is unsatisfied. Following this logic we can show that the set of people who point at 1 under R is a subset of the set of people who point at 1 under R'. Moreover, trading cycles that do not involve 1 are the same under R and R'. In this case, 7 points at 8 and 8 points at 7. Therefore, only trades that involve 1 under R might not happen under R'. Since under R' 1 trades in Step 1, the Pre-trade Claim does not say anything about Step 2. •

If $t' \leq t$, by *pre-trade inclusion*, and the $\underline{t}$ *equality* lemma, $O_t \subseteq O'_t$ and $O'_t \subseteq O'_{t'}$. Thus, $\alpha(i) \in O'_{t'}$. Since i is part of a *trading cycle* at Step t' and by definition of R', i points in Step t' at whoever holds $\alpha(i)$ at $\underline{t}$. Then, i is assigned one of his most preferred objects in $O'_{t'}$ which is uniquely $\alpha(i)$. Thus, $\alpha'(i) = \alpha(i)$.

We only need to show that $\alpha(i) = \alpha'(i)$ when $t' > t$. We first state the following claim.

Claim 2. ***(Post-trade inclusion)*** *For each $\ddot{t} \in \{t.., t'\}$,*

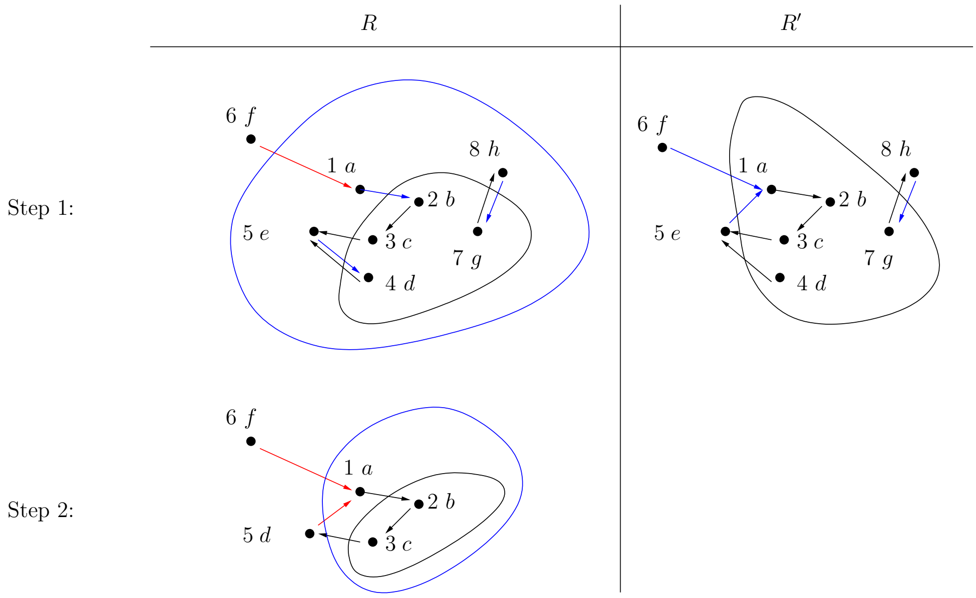


Figure 6: Example of the Pre-trade Claim. If 1 reports R'_1 rather than R_1, the set of people who point at 1 has under R' is a superset of those who point at him under R. Moreover, trades that do not involve 1 that are realized under R are also realized under R'.

$$\textit{(i)}\quad \begin{array}{ll} O_{\ddot t} \subseteq O'_{\ddot t}, & N_{\ddot t} \subseteq N'_{\ddot t}, \\ O'_{\ddot t} \setminus O_{\ddot t} \subseteq h_{\ddot t}(CONN(i, R', \ddot t - 1)),\ \textit{and} & N'_{\ddot t} \setminus N_{\ddot t} \subseteq CONN(i, R', \ddot t - 1) \end{array},$$

(ii) $S'_{\ddot t} \subseteq S_{\ddot t}$ *and* $S'_{\ddot t} \setminus S_{\ddot t} \subseteq CONN(i, R', \ddot t - 1)$,

(iii) *For each* $j \in N'_{\ddot t} \setminus CONN(i, R', \ddot t)$, $p_{\ddot t}(j) = p'_{\ddot t}(j)$, *and*

(iv) *For each* $j \in N'_{\ddot t} \setminus CONN(i, R', \ddot t)$, $h_{\ddot t+1}(j) = h'_{\ddot t+1}(j)$.

The proof of this claim is similar to that of *pre-trade inclusion* and is available upon request. Example 6 should help the reader build some intuition with regards to Claim 2.

Example 6. *Post-trade Claim.*

Let $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $\omega = (a, b, c, d, e, f, g, h)$, and $R \in \mathcal{R}^N$ be as follows.

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R'_1
$b\ c$	$a\ d$	g	$a\ c\ d\ e$	$a\ d\ e\ f$	b	$d\ g\ h$	g	c
		$b\ d$						$a\ b$

Like the Pre-trade Claim, the Post-trade Claim deals with situations where 1 reports R'_1, a *local push-up* of R_1 at his assignment $TC^{\prec}(R,\omega)(1) = c$, rather than R_1 (figure 7.) It provides us with a description of each step between the step at which 1 trades under R and the step that he trades under $R'(\equiv (R'_1, R_{-1}))$. The Post-trade Claim says that before 1 trades under R', the set of people who point at 1 under R is a subset of those who point at 1 under R'. Moreover, trades that do not involve 1 that occur under R also occur under R'. Therefore, the people that trade under R is a superset of those who trade under R'. Finally, since there is less trade under R, the sets of objects and people remaining at a particular step under R are subsets of the objects and people remaining at the same step under R'.

The claim does not say anything about Step 1 because 1 has not traded yet. By the $\underline{t}$-equality lemma, we know that the departure stage of Step 1 is the same under R and R' and in the pointing stage, everyone but 1 points in the same way under R and R'. In this example, 1 trades under R but not under R'. In Step 2, the departure phase is the same. In the pointing stage, since 1 is satisfied under R but not under R', 1 has more people who point at him under R'. Under R', 4 and 5 point at 1. Therefore, the set of people who point at 1 under R is a subset of those who point at 1 under R'. Even though it is not shown in this example, using the same reasoning as in the Pre-trade Claim, trading cycles that do not involve 1 that occur under R also occur under R' and at least as many people trade under R as under R'. Since under R' 1 trades in Step 2, the Post-trade Claim does not say anything about Step 3. •

Suppose $\alpha'(i) \neq \alpha(i)$. Since i is assigned $\alpha(i)$ under R, there is $\tilde{t}$ such that $h_{\tilde{t}+1}(i) = \alpha(i)$. By *post-trade inclusion*, $\tilde{t} < t'$. Since $\tau(R'_i, O_{\tilde{t}}) = \{\alpha(i)\}$, we have $i \xrightarrow[\tilde{t}]{R'} j \in N'_{\tilde{t}}$ such that $h'_{\tilde{t}}(j) = \alpha(i)$. Since $\alpha'(i) \neq \alpha(i)$, $h'_{\tilde{t}+1}(i) \neq \alpha(i)$. Thus $j \notin CONN(i, R', \tilde{t}+1)$. Thus by *post-trade inclusion*, $h_{\tilde{t}}(j) = h'_{\tilde{t}}(j) = \alpha(i)$. Since $j \notin CONN(i, R', \tilde{t})$, we have $j \xrightarrow[\tilde{t}]{R'} j_1(\neq i) \xrightarrow[\tilde{t}]{R'} j_2(\neq i) \ldots \xrightarrow[\tilde{t}]{R'} j_r(\neq i)$. Again, by *post-trade inclusion*, $j \xrightarrow[\tilde{t}]{R} j_1(\neq i) \xrightarrow[\tilde{t}]{R} j_2(\neq i) \ldots \xrightarrow[\tilde{t}]{R} j_r(\neq i)$. This contradicts $h_{\tilde{t}}(i) = \alpha(i)$. ♣

We are now ready to show that $TC^{\prec}$ is *strategy-proof.*

Proposition 5. *For priority order $\prec$, $TC^{\prec}(R,\omega)$ is strategy-proof.*

Proof: Suppose that $TC^{\prec}$ is not *strategy-proof.* Then, there is $(R,\omega) \in \mathcal{R}^N \times A$, $i \in N$ and $R'_i \in \mathcal{R}$ such that $TC^{\prec}(R'_i, R_{-i}, \omega)(i)\ P_i\ TC^{\prec}(R,\omega)(i)$. Let $\alpha \equiv TC^{\prec}(R,\omega)$ and $\alpha' \equiv TC^{\prec}(R'_i, R_{-i}, \omega)$. By the *invariance* lemma, we only need to

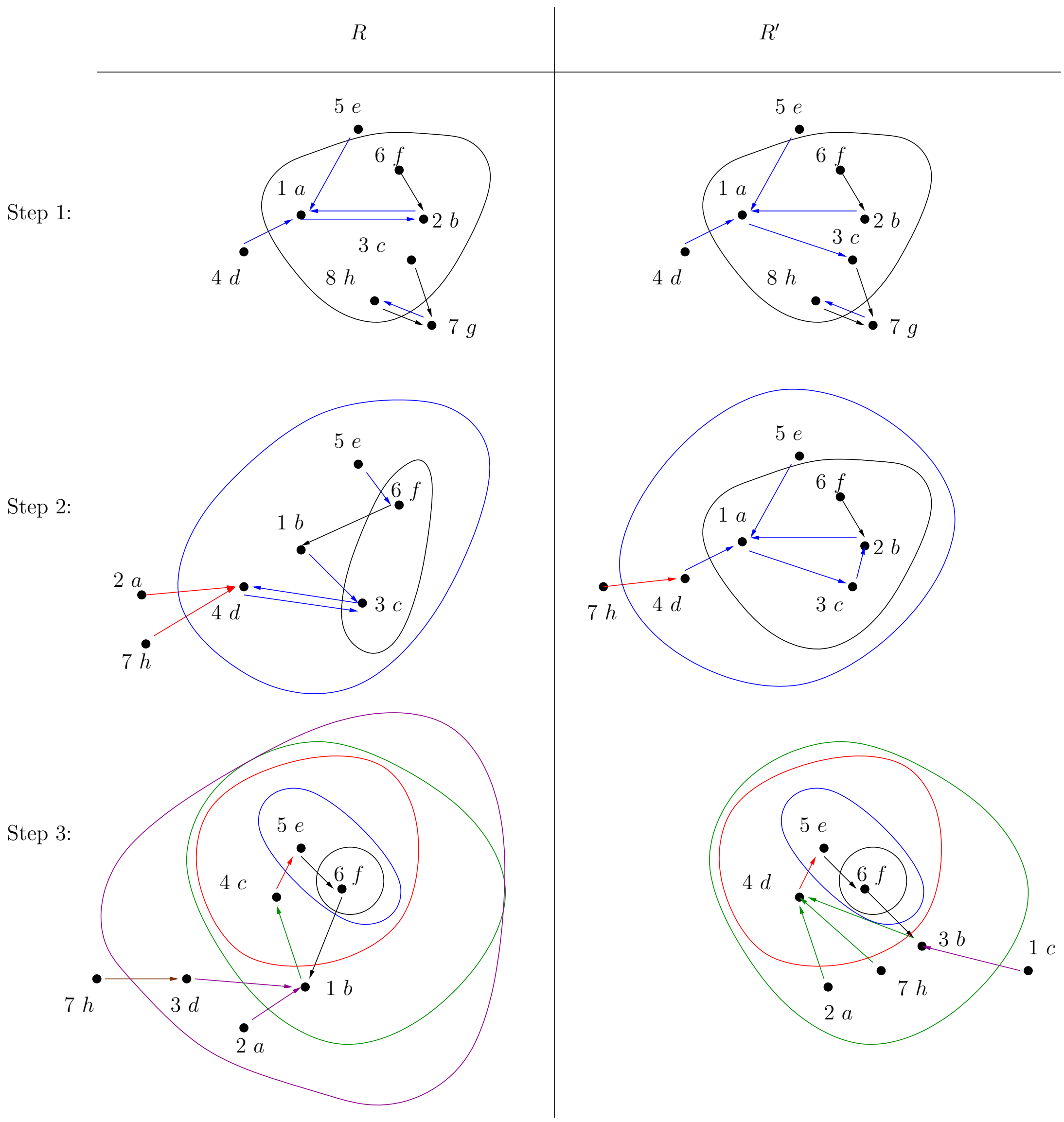


Figure 7: Example of the Post-trade Claim. If 1 reports R_1' rather than R_1, the set of people who point at him has under R' includes those who point at him under R. Moreover, trades that do not involve 1 are realized under R are also realized under R'.

consider R'_i such that $I(\alpha'(i), R_i) = \{\alpha'(i)\}$. Otherwise, there is $R_i^{\alpha'(i)\uparrow} \in R$ such that $TC^{\prec}(R_i^{\alpha(i)\uparrow}, R_{-i}, \omega)(i) = \alpha'(i)$ and thus, $TC^{\prec}(R_i^{\alpha'(i)\uparrow}, R_{-i}, \omega)(i)\ P_i\ \alpha(i)$.

Define $t, t', \bar{t}$, and $\underline{t}$ as in the proof of the *invariance* lemma. Since $\alpha'(i) \neq \omega(i), \alpha'(i)\ P'_i\ \omega(i)$ and for each $\ddot{t} \leq t'$, $i \in U'_{\ddot{t}}$. We consider the following cases.

Case 1: $\underline{t} = \bar{t} \leq t'$. In this case, $i \in S_{\bar{t}}$. That is, $\omega(i) \in \tau(R_i, O_{\bar{t}})$. By the $\underline{t}$ *equality* lemma, $O_{\bar{t}} = O'_{\underline{t}}$. Since $\alpha'(i) \in O'_{\bar{t}}$, $\alpha'(i) \in O_{\bar{t}}$. Thus, $\omega(i)\ R_i\ \alpha'(i)$ and by *individual rationality*, $\alpha(i)\ R_i\ \alpha'(i)$.

Case 2: $\underline{t} = t' < t$. By the $\underline{t}$ *equality* lemma, $O_{t'} = O'_{t'}$, $N_{t'} = N'_{t'}$, and for each $j \in N'_{t'} \setminus \{i\}$, $p_{t'}(j) = p'_{t'}(j)$ and $h_{t'}(j) = h'_{t'}(j)$. Since i trades under R' at t' and by definition of R', $I(\alpha'(i), R_i) = \{\alpha'(i)\}$, then $\{h'_{t'+1}(i)\} = \{\alpha'(i)\} = \tau(R'_i, O'_{t'})$. Then, i leaves with $\alpha'(i)$. Therefore, there is $\{j_1, j_2, \ldots, j_3\} \subseteq N'_{\underline{t}}$ such that $j_1 \xrightarrow[\underline{t}]{R'} j_2 \xrightarrow[\underline{t}]{R'} j_3 \ldots \xrightarrow[\underline{t}]{R'} j_r \xrightarrow[\underline{t}]{R'} i$ and $h'_{\underline{t}}(j_1) = \alpha'(i)$. Then, by the $\underline{t}$ *equality* lemma, $j_1 \xrightarrow[\underline{t}]{R} j_2 \xrightarrow[\underline{t}]{R} j_3 \ldots \xrightarrow[\underline{t}]{R} j_r \xrightarrow[\underline{t}]{R} i$ and $h_{\underline{t}}(j_1) = \alpha'(i)$. By *persistence*, $h_{t+1}(i)\ R_i\ \alpha'(i)$.

Case 3: $\underline{t} = t \leq t'$. Since $h_{t+1}(i) \in \tau(R_i, O_t)$ and $h_{t+1}(i)\ I_i\ \alpha(i)$, $\alpha(i) \in \tau(R_i, O_t)$. Since $\alpha'(i) \in O'_t$ and by the $\underline{t}$ *equality* lemma $O'_t = O_t$ we have $\alpha'(i) \in O_t$. Thus, $\alpha(i)\ R_i\ \alpha'(i)$. ■

Appendices

A A formal definition of the top cycles algorithm with fixed tie breaking

For $t = 0, 1, 2, \ldots$, in Step t, we define the $\boldsymbol{O_t} \subsetneq \boldsymbol{O}$, the $\boldsymbol{N_t} \subsetneq \boldsymbol{N}$, and $\boldsymbol{h_{t+1} : N_t \to O_t}$. We also define, for each $i \in N_t$, the $\boldsymbol{p_t(i)}$.

Let $O_0 \equiv O$, $N_0 \equiv N$, and $h_1 \equiv \omega$.

At step $t = 1, 2, \ldots$, we get (O_t, N_t, h_{t+1}, p_t) as follows.

Departure phase: To determine who leaves we use an iterative procedure. Let G^1_t be the *largest* group in N_{t-1} such that:

i) What each $i \in G^1_t$ holds is among his most preferred objects:

$$h_t(i) \in \tau(R_i, h_t(N_{t-1}))$$

ii) The most preferred objects of the group are hold by them:

$$\tau(R_i, h_t(N_{t-1})) \subseteq h_t(G^1_t)$$

Among the remaining people, $N_{t-1} \setminus G_t^1$ we determine G_t^2 using the same conditions. We continue until we find $K \in \{1, ..., |N_{t-1}|\}$ such that $G_t^K = \varnothing$ and for each $k < K$, $G_t^k \neq \varnothing$. Then, each $i \in \cup_{k=1}^K G_t^k$, departs with $h_t(i)$. That is, $TC^{\prec}(R, \omega)(i) \equiv h_t(i)$. Further,

$$\begin{aligned} N_t &\equiv N_{t-1} \setminus \cup_{k=1}^K G_t^k \text{ and} \\ O_t &\equiv h_t(N_t).^{17} \end{aligned}$$

Note that the if $K = 1$ the set of people departing is empty, then $N_t = N_{t-1}$ and $O_t = O_{t-1}$.

Pointing phase: We determine p_t in stages, as follows: For each $i \in N_t$, let i's **candidate pointees, $\boldsymbol{C_{i,t}}$** $\equiv \{j \in N_t : h_t(j) \in \tau(R_i, O_t)\}$, be the people that hold one of i's most preferred objects.

Stage 1) If $t \neq 1$, we first consider $i \in N_t$ such that i's pointee in Step $t-1$ has not departed and holds the same object as he did at Step $t-1$. Then, i points at the same person in Step t as well. That is, if $t \neq 1$, for each $i \in N_t$ such that $i \underset{t-1}{\longrightarrow} j \in N_t$, and $h_t(j) = h_{t-1}(j)$, we have $i \underset{t}{\longrightarrow} j$.

Stage 2) We consider $i \in N_t$ that has only one candidate pointee. He points at his unique candidate pointee. That is, for each $i \in N_t$ such that $C_{i,t} = \{j\}$, we have $i \underset{t}{\longrightarrow} j$.

Stage 3) We consider $i \in N_t$ with at least one unsatisfied candidate pointee. He points at the unsatisfied candidate pointee with highest priority.[18] That is,

$$p_t(i) \equiv \arg \underset{j \in C_{i,t} \setminus S_t}{\prec\text{-max}}\, j.^{19}$$

Stage 4) We consider $i \in N_t$ with only satisfied candidate pointees, at least one of whom has an unsatisfied pointee, $C_{i,t}^1 \equiv \{j \in C_{i,t} : j \underset{t}{\longrightarrow} U_t\} \subseteq S_t$. Then i points at the satisfied candidate whose unsatisfied pointee has highest priority (breaking ties with respect to $\prec$). That is, $p_t(i) = J_t(i)$, where $J_t(i) \equiv \arg \underset{j \in C_{i,t}^1}{\prec\text{-max}}\, p_t(j)$ and $|J_t(i)| = 1$. In case two or more

[18] The order within a stage is unimportant. In addition, since stages are performed sequentially, if $p_t(i)$ is defined at Stage k, then $p_t(p_t(i))$ is defined at Stage $k' < k$. Further, $p_t(i)$ is independent of $p_t(j)$ if $p_t(j)$ is defined at Stage $k'' \geq k$.

[19] We define $\arg \underset{j \in C_{i,t} \setminus S_t}{\prec\text{-max}}\, j$ as the person that maximizes the priority $\prec$. In general, for each $f : X \to N$ and each $X' \subseteq X$, we define $\arg \underset{j \in X'}{\prec\text{-max}}\, f(j) \equiv i \in X'$ such that for each $j \in X' \setminus \{i\}$, $f(i) \prec f(j)$.

of these satisfied candidates points at the unsatisfied pointee with the highest priority ($|J_t(i)| > 1$), i points at the satisfied candidate with the highest priority. That is, $p_t(i) \equiv \arg \underset{j \in J_t(i)}{\prec\text{-max}}\, j$.

Stage 5) We consider $i \in N_t$ whose candidate pointees are all satisfied and have satisfied pointees, at least one of whom has an unsatisfied pointee. He points at the candidate who points at the person who points at the unsatisfied person with highest priority (again, breaking ties with $\prec$). That is,

$$\begin{aligned} C_{i,t}^2 &\equiv \{j \in C_{i,t} : j \underset{t}{\longrightarrow} \underset{t}{\longrightarrow} U_t\} \subseteq S_t, \\ J_t(i) &\equiv \arg \underset{j \in C_{i,t}^2}{\prec\text{-max}}\, p_t(p_t(j)), \text{ and} \\ p_t(i) &\equiv \arg \underset{j \in J_t(i)}{\prec\text{-max}}\, j. \end{aligned}$$

Stage ...) The process is repeated until for each $i \in N_t$, $p_t(i)$ is defined.

By definition of the departure phase, each $i \in N_t$ points, directly or indirectly, at an unsatisfied person.[20] Thus, the pointing phase terminates in a finite number of stages.

Trading phase: There is at least one cycle $C \equiv \{i_1, i_2, \ldots, i_s\}$ such that $i_1 \underset{t}{\longrightarrow} i_2 \underset{t}{\longrightarrow} \ldots \underset{t}{\longrightarrow} i_s \underset{t}{\longrightarrow} i_1$. Further, each $i \in N_t$ is a member of at most one cycle. We get h_{t+1} by performing the trades prescribed by each cycle. That is, for each cycle, $\{i_1, i_2, \ldots, i_s\}$, and each $k = 1, \ldots s$, $h_{t+1}(i_{k-1}) = h_t(i_k)$. For each $i \in N_t$ who is not in a cycle, $h_{t+1}(i) = h_t(i)$.

The algorithm terminates at Step $\mathring{t}$ such that $N_{\mathring{t}} = \varnothing$. △

B Proof of the $\underline{t}$-equality Lemma

Proof of Lemma 1: Note that if in Step 1, $i \in S_1$ and $i \in S_1'$, the statement of this Lemma is vacuously satisfied. Then, assume $i \in U_1$ and $i \in U_1'$.

Step 1: Since $i \in U_1$ and $i \in U_1'$, and for each $j \in N \setminus \{i\}$, $R_j = R_j'$ and $h_1(j) = h_1'(j) = \omega(j)$, we have $S_0 = S_0'$. Thus, $O_1 = O_1'$ and $N_1 = N_1'$. Therefore, for each $j \in N_1 \setminus \{i\}$, $p_1(j) = p_1'(j)$.

If $1 < \underline{t}$, i does not trade at Step 1 under either R or R'. Therefore, the cycles formed under p_1 and p_1' are the same and do not involve i. Then, for each $j \in N_1$, $h_2(j) = h_2'(j)$.

[20] This condition implies that each cycle includes at least one unsatisfied person.

Step 2: Since $i \in U_1$ and $i \in U_1'$, and for each $j \in N_1 \setminus \{i\}, R_j|_{O_1} = R_j'|_{O_1}$ and $h_2(j) = h_2'(j)$, we have $S_1 = S_1'$. Thus, $O_2 = O_2'$ and $N_2 = N_2'$.

As an **induction hypothesis**, suppose that for some $\ddot{t} < \underline{t}$, $O_{\ddot{t}} = O'_{\ddot{t}}$, $N_{\ddot{t}} = N'_{\ddot{t}}$, $h_{\ddot{t}} = h'_{\ddot{t}}$, and for each $j \in N_{\ddot{t}} \setminus \{i\}$, $p_{\ddot{t}-1}(j) = p'_{\ddot{t}-1}(j)$.

Step $\ddot{t}+1$: We show that for $\ddot{t} < \underline{t}$, $O_{\ddot{t}+1} = O'_{\ddot{t}+1}$, $N_{\ddot{t}+1} = N'_{\ddot{t}+1}$, $h_{\ddot{t}+1} = h'_{\ddot{t}+1}$, and for each $j \in N_{\ddot{t}} \setminus \{i\}$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$.

Since $\ddot{t} < \underline{t}$, $i \in U_{\ddot{t}}$ and $i \in U'_{\ddot{t}}$. In addition, by our *induction hypothesis*, $O_{\ddot{t}} = O'_{\ddot{t}}$, $N_{\ddot{t}} = N'_{\ddot{t}}$, $h_{\ddot{t}} = h'_{\ddot{t}}$ and $R_j|_{O_{\ddot{t}}} = R'_j|_{O_{\ddot{t}}}$. Thus, for each $j \in N_{\ddot{t}} \setminus \{i\}$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$.

Since $\ddot{t} < \underline{t}$, i does not trade under R or R' at $\ddot{t}$. Therefore, the cycles formed by $p_{\ddot{t}}$ and $p'_{\ddot{t}}$ are the same and do not involve i. Thus, for each $j \in N_{\ddot{t}}$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$. Also, $O_{\ddot{t}+1} = O'_{\ddot{t}+1}$ and $N_{\ddot{t}+1} = N'_{\ddot{t}+1}$. ♣

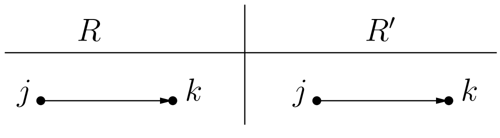
C Proof of the pre-trade inclusion claim

Proof of Claim 6: Suppose $\underline{t} \neq \min\{t, t'\}$. Then $\underline{t} = \bar{t}$. Since $\tau(R_i', O_{\bar{t}}) = \{\alpha(i)\}$, $\underline{t} < t'$, and by definition of $\bar{t}$, $i \in U'_{\bar{t}}$ and $i \in S_{\bar{t}}$.

Let $\ddot{t} = \bar{t}$. Statements (i) and (ii), for $\bar{t}$, are implied by the $\underline{t}$ *equality* lemma. Further, $S_{\bar{t}} = S'_{\bar{t}} \cup \{i\}$.

We now prove statement (iii), for $\bar{t}$, by following the progression of the pointing phase.[21] By the $\underline{t}$ *equality* lemma, each $j \in N_{\bar{t}} \setminus \{i\}$ pointed at the same person under R as he did under R' at step $\bar{t}-1$.

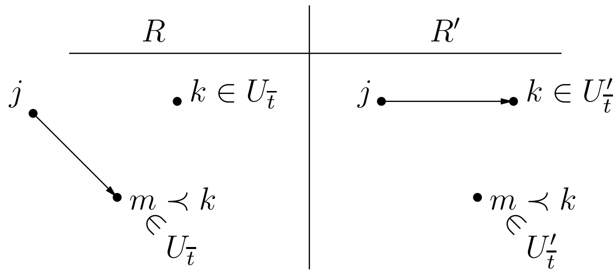
Stage 1) At the beginning of the pointing phase we consider people who were pointing at someone who remains in $N_{\bar{t}}$ and holds the same object. In particular, we consider $j \in N'_{\bar{t}} \setminus CONN(i, R', \bar{t})$ such that $j \xrightarrow[\bar{t}-1]{R'} k \in N'_{\bar{t}}$ and $h'_{\bar{t}}(k) = h'_{\bar{t}-1}(k)$. Then, $j \xrightarrow[\bar{t}]{R'} k$. By the $\underline{t}$ *equality* lemma, $j \xrightarrow[\bar{t}-1]{R} k$ and $h_{\bar{t}}(k) = h_{\bar{t}-1}(k) = h'_{\bar{t}-1}(k)$. Thus $j \xrightarrow[\bar{t}]{R} k$.



[21]We provide a graphical illustration of the argument following each stage of the pointing phase.

Stage 2) Now we consider people who have a unique most preferred object. They point at the same person under R as under R'.

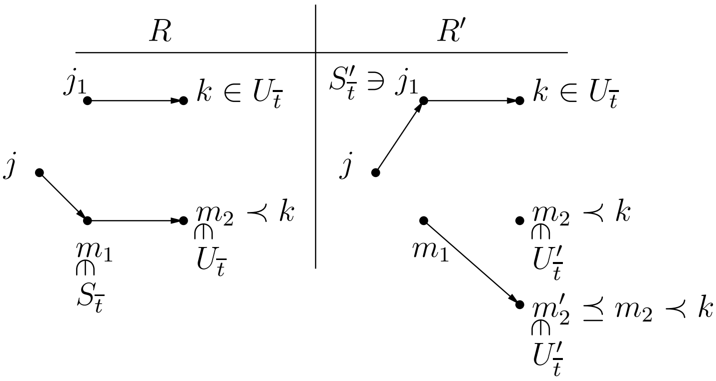
Stage 3) Next, we consider the people who point at unsatisfied people under R'. In particular, $j \in N'_{\bar{t}} \setminus CONN(i, R', \bar{t})$ such that $j \xrightarrow[\bar{t}]{R'} k \in U'_{\bar{t}}$. Since $j \notin CONN(i, R', \bar{t})$, $k \notin CONN(i, R', \bar{t})$. Since $k \in U'_{\bar{t}}$ and $S_{\bar{t}} = S'_{\bar{t}} \cup \{i\}$, $k \in U_{\bar{t}}$. Further, $h_{\bar{t}}(k) = h'_{\bar{t}}(k) = \omega(k)$. Suppose $j \xrightarrow[\bar{t}]{R} m \neq k$. Then, $m \in U_{\bar{t}} \subseteq U'_{\bar{t}}$ and so $h_{\bar{t}}(m) = h'_{\bar{t}}(m) = \omega(m)$ and $m \prec k$. This contradicts $j \xrightarrow[\bar{t}]{R'} k$.



Stage 4) We now consider the people who point at satisfied people with unsatisfied pointees, under R'. In particular, we consider $j \in N'_{\bar{t}} \setminus CONN(i, R', \bar{t})$ such that $j \xrightarrow[\bar{t}]{R'} j_1 \in S'_{\bar{t}} \xrightarrow[\bar{t}]{R'} k \in U'_{\bar{t}}$. Then, by (ii), $j_1 \in S_{\bar{t}}$.

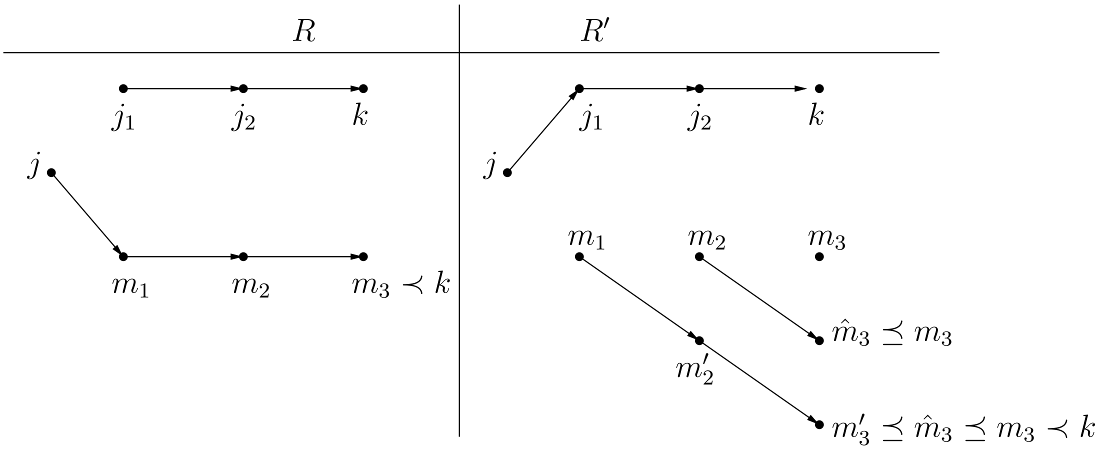
By the preceding arguments, $j_1 \xrightarrow[\bar{t}]{R} k$ and $k \in U_{\bar{t}}$. Suppose $j \xrightarrow[\bar{t}]{R} m_1 \neq j_1$. If $m_1 \in U_{\bar{t}}$, then $h_{\bar{t}}(m_1) = h'_{\bar{t}}(m_1) = \omega(m_1)$ and $m_1 \in U'_{\bar{t}}$. But this contradicts $j \xrightarrow[\bar{t}]{R'} S'_{\bar{t}}$. So $m_1 \in S_{\bar{t}}$ and $m_1 \xrightarrow[\bar{t}]{R} m_2$ such that $m_2 \in U_{\bar{t}}$ and $m_2 \prec k$. Then, $m_2 \in U'_{\bar{t}}$ and thus $m_1 \xrightarrow[\bar{t}]{R'} m'_2 \in U'_{\bar{t}}$ such that $m'_2 \preceq m_2 \prec k$. [22] By the $\underline{t}$ *equality* lemma, $h_{\bar{t}}(m_1) = h'_{\bar{t}}(m_1)$. This contradicts $j \xrightarrow[\bar{t}]{R'} j_1$.

[22]We use the notation $i \preceq j$ to indicate $i \prec j$ or $i = j$.



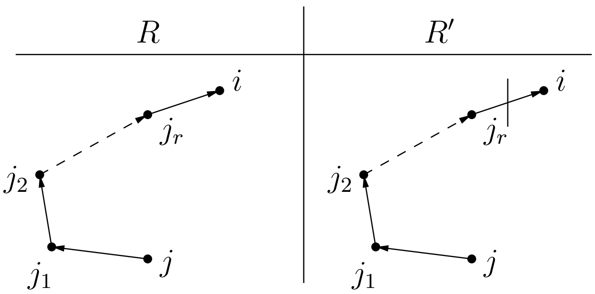
Stage 5) Now we consider the people who point at satisfied people with satisfied pointees whose pointees are unsatisfied, under R'. Particularly, consider $j \in N'_{\bar t} \setminus CONN(i, R', \bar t)$ be such that $j \xrightarrow[\bar t]{R'} j_1 \in S'_{\bar t} \xrightarrow[\bar t]{R'} j_2 \in S'_{\bar t} \xrightarrow[\bar t]{R'} k \in U'_{\bar t}$. Then, $j_1, j_2 \in S_{\bar t}$.

By the preceding arguments, $j_1 \xrightarrow[\bar t]{R} j_2 \xrightarrow[\bar t]{R} k \in U_{\bar t}$. Suppose $j \xrightarrow[\bar t]{R} m_1 \neq j_1$. If $m_1 \in U_{\bar t}$, then $h_{\bar t}(m_1) = h'_{\bar t}(m_1) = \omega(m_1)$ and $m_1 \in U'_{\bar t}$. But this contradicts $j \xrightarrow[\bar t]{R'} S'_{\bar t}$. So $m_1 \in S_{\bar t}$. By the $\underline{t}$ *equality* lemma, $h_{\bar t}(m_1) = h'_{\bar t}(m_1)$. Let $m_1 \xrightarrow[\bar t]{R} m_2$. If $m_2 \in U_{\bar t}$, then $h_{\bar t}(m_2) = h'_{\bar t}(m_2) = \omega(m_2)$ and $m_2 \in U'_{\bar t}$. So $m_1 \xrightarrow[\bar t]{R'} U'_{\bar t}$. But this contradicts $j \xrightarrow[\bar t]{R'} S'_{\bar t} \xrightarrow[\bar t]{R'} S'_{\bar t}$. So $m_2 \in S_{\bar t}$. By the $\underline{t}$ *equality* lemma, $h_{\bar t}(m_2) = h'_{\bar t}(m_2)$. Since $k \in U_{\bar t}$, $m_2 \xrightarrow[\bar t]{R} m_3 \in U_{\bar t}$ and $m_3 \prec k$. Then, $m_3 \in U'_{\bar t}$ and so $m_2 \xrightarrow[\bar t]{R'} \hat m_3 \in U'_{\bar t}$ such that $\hat m_3 \preceq m_3 \prec k$. If $m_1 \xrightarrow[\bar t]{R'} m_2$, this contradicts $j \xrightarrow[\bar t]{R'} j_1$. Then $m_1 \xrightarrow[\bar t]{R'} m'_2 \neq m_2$ and $m'_2 \xrightarrow[\bar t]{R'} m'_3$. Note that $m'_2 \in S'_{\bar t}$, otherwise this contradicts $j \xrightarrow[\bar t]{R'} S'_{\bar t} \xrightarrow[\bar t]{R'} S'_{\bar t}$. In addition, since $m_1 \not\xrightarrow[\bar t]{R'} m_2$ and $m_1 \xrightarrow[\bar t]{R'} m'_2$, we have $m'_3 \in U'_{\bar t}$ and $m'_3 \prec \hat m_3$. Then, $m_3 \prec k$, which contradicts $j \xrightarrow[\bar t]{R'} j_1$.



Stage ...) Repeating this argument for the rest of the pointing phase, we show (iii).

We show that (v) $CONN(i, R, \bar{t}) \subseteq CONN(i, R', \bar{t})$ is a consequence of (iii). To see this, suppose $j \in CONN(i, R, \bar{t}) \setminus CONN(i, R', \bar{t})$. Then, there is a sequence $\{j_1, j_2, ..., j_r, i\} \subset N_{\bar{t}}$, such that $j \xrightarrow[\bar{t}]{R} j_1 \xrightarrow[\bar{t}]{R} j_2 \xrightarrow[\bar{t}]{R} ... \xrightarrow[\bar{t}]{R} j_r \xrightarrow[\bar{t}]{R} i$. Since $j \notin CONN(i, R', \bar{t})$, then by (iii), $j \xrightarrow[\bar{t}]{R'} j_1$. Then, $j_1 \notin CONN(i, R', \bar{t})$. Again, by (iii), $j_1 \xrightarrow[\bar{t}]{R'} j_2$ and $j_2 \notin CONN(i, R', \bar{t})$. Repeating the argument r times, $j_r \notin CONN(i, R', \bar{t})$. By (iii), $j_r \xrightarrow[\bar{t}]{R'} i$, and this contradicts $j \notin CONN(i, R', \bar{t})$.



Finally, we prove (iv) for Step $\bar{t}$. We show that for each $j \in N'_{\bar{t}} \setminus CONN(i, R', \bar{t})$, $h_{\bar{t}+1}(j) = h'_{\bar{t}+1}(j)$. Note that since at $\bar{t} < t'$, i does not trade. Then, no *trading cycle* under R' involves i. So, no *trading cycle* involves any member of $CONN(i, R', \bar{t})$. That is, for each *trading cycle* $C' \subset N'_{\bar{t}}$, $CONN(i, R', \bar{t}) \cap C' = \varnothing$. By (iii) and since $N_{\bar{t}} = N'_{t''}$, we have $C' \subset N_{\bar{t}}$ is also a *trading cycle* under R. Therefore, for each $j \in N'_{\bar{t}} \setminus CONN(i, R', \bar{t})$, $h'_{\bar{t}+1}(j) = h_{\bar{t}+1}(j)$. Moreover, for each $j \in CONN(i, R', \bar{t})$, $h'_{\bar{t}+1}(j) = h'_{\bar{t}}(j)$.

As an **induction hypothesis**, suppose that for some $\ddot{t} \in \{\underline{t}, ..., \min\{t, t'\} - 1\}$,

(i) $O_{\ddot{t}} \subseteq O'_{\ddot{t}}$, $O'_{\ddot{t}} \setminus O_{\ddot{t}} \subseteq h_{\ddot{t}}(CONN(i, R', \ddot{t} - 1))$, and $N_{\ddot{t}} \subseteq N'_{\ddot{t}}$ $N'_{\ddot{t}} \setminus N_{\ddot{t}} \subseteq CONN(i, R', \ddot{t} - 1)$,

(ii) $S'_{\ddot{t}} \subseteq S_{\ddot{t}}$ and $S'_{\ddot{t}} \setminus S_{\ddot{t}} \subseteq CONN(i, R', \ddot{t} - 1)$,

(iii) For each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $p_{\ddot{t}}(j) = p'_{\ddot{t}}(j)$,

(iv) For each $j \in N'_{\ddot{t}} \setminus CONN(i, R', \ddot{t})$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$, and

(v) $CONN(i, R, \ddot{t}) \subseteq CONN(i, R', \ddot{t})$.

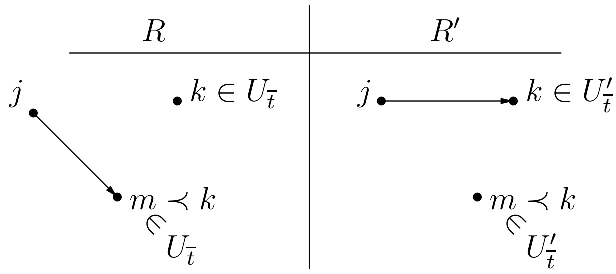
We prove that these statements are true of $\ddot{t}+1$. To prove (i) and (ii) for $\ddot{t}+1$, note that by (iv) and (v) of the *induction hypothesis*, if $C \in N_{\ddot{t}}$ is a *trading cycle* under R and is not a *trading cycle* under R', then $C \subseteq CONN(i, R', \ddot{t})$. Thus, at Step $\ddot{t} + 1$, we have statements (i) and (ii).

We now prove (iii), for $\ddot{t}+1$, by following the progression of the pointing phase just as in the case of $\bar{t}$.

Stage 1) We consider people whose pointee at $\ddot{t}$ remains at $\ddot{t} + 1$ and holds the same object under R as R'. In particular, we consider $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ such that $j \xrightarrow[\ddot{t}]{R'} k \in N'_{\ddot{t}}$ and $h'_{\ddot{t}+1}(k) = h'_{\ddot{t}}(k)$. Then, $j \xrightarrow[\ddot{t}+1]{R'} k$. By the *induction hypothesis*, $j \xrightarrow[\ddot{t}]{R} k$ and $h_{\ddot{t}+1}(k) = h_{\ddot{t}}(k) = h'_{\ddot{t}}(k)$. Thus $j \xrightarrow[\ddot{t}+1]{R} k$.

Stage 2) Now we consider people who have a unique most preferred object. For each $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t} + 1)$, if $\tau(R_j, O'_{\ddot{t}+1}) = \{a\}$, then by the *induction hypothesis*, $h^{-1}_{\ddot{t}+1}(a) = h'^{-1}_{\ddot{t}+1}(a) \notin CONN(i, R', \ddot{t} + 1)$. Thus, $a \in O_{\ddot{t}+1}$ and so $p_{\ddot{t}+1}(j) = p'_{\ddot{t}+1}(j)$.

Stage 3) Next, we consider the people with unsatisfied pointees under R'. In particular, $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t} + 1)$ such that $j \xrightarrow[\ddot{t}+1]{R'} k \in U'_{\ddot{t}}$. Since $j \notin CONN(i, R', \ddot{t}+1)$, $k \notin CONN(i, R', \ddot{t}+1)$. Since $k \in U'_{\ddot{t}+1}$ and $S_{\ddot{t}+1} \setminus S'_{\ddot{t}+1} \subseteq CONN(i, R', \ddot{t} + 1)$, $k \in U_{\ddot{t}+1}$. Further, $h_{\ddot{t}+1}(k) = h'_{\ddot{t}+1}(k) = \omega(k)$. Suppose $j \xrightarrow[\ddot{t}+1]{R} m \neq k$. Then, $m \in U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$ and so $h_{\ddot{t}+1}(m) = h'_{\ddot{t}+1}(m) = \omega(m)$ and $m \prec k$. This contradicts $j \xrightarrow[\ddot{t}+1]{R'} k$.

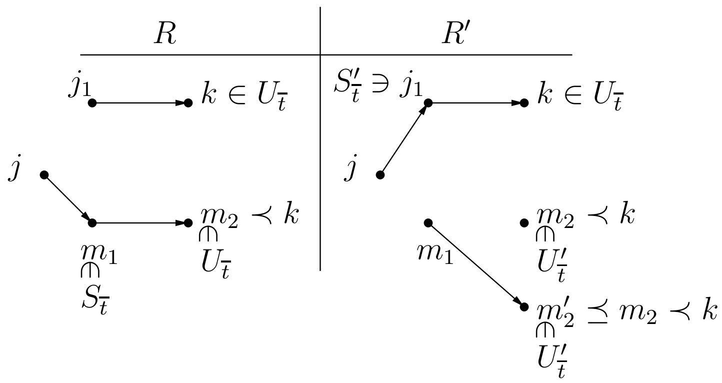


Stage 4) We now consider the people who point at satisfied people with unsatisfied pointees, under R'. In particular, we consider $j \in N'_{\ddot t+1} \setminus CONN(i, R', \ddot t+1)$ such that $j \xrightarrow[\ddot t+1]{R'} j_1 \in S'_{\ddot t+1} \xrightarrow[\ddot t+1]{R'} k \in U'_{\ddot t+1}$. Then, by (ii), $j_1 \in S_{\ddot t+1}$.

By the preceding arguments, $j_1 \xrightarrow[\ddot t+1]{R} k$ and $k \in U_{\ddot t+1}$. Suppose $j \xrightarrow[\ddot t+1]{R} m_1 \neq j_1$. We consider the following two cases.

$h'_{\ddot t+1}(m_1)$
||
$h_{\ddot t+1}(m_1)$: If $m_1 \in U_{\ddot t+1}$, then $m_1 \in U'_{\ddot t+1}$ and $h_{\ddot t+1}(m_1) = h'_{\ddot t+1}(m_1) = \omega(m_1)$.

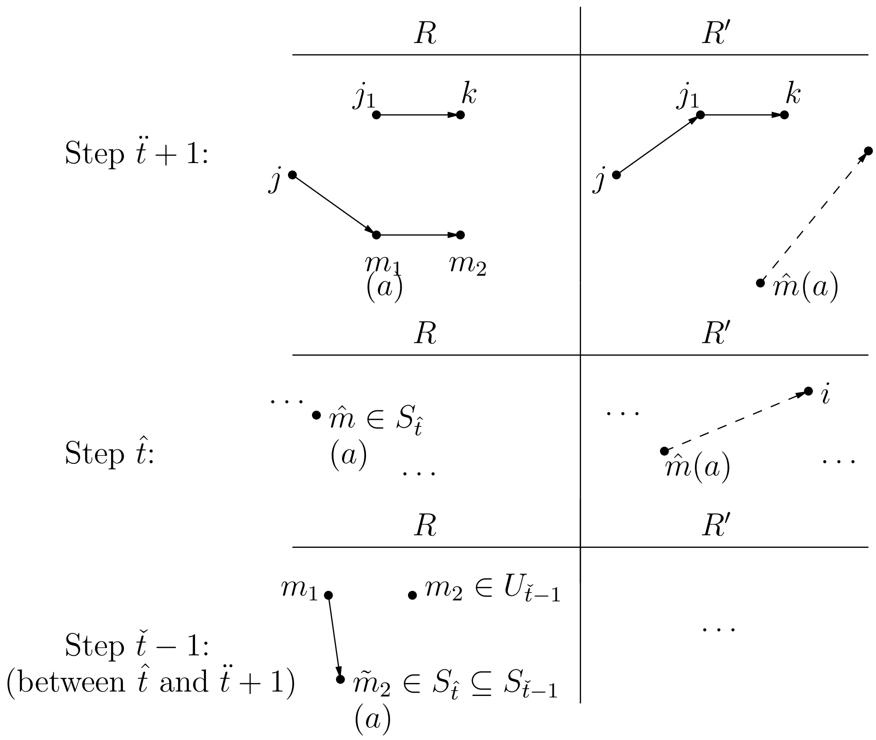
Then, $j \xrightarrow[\ddot t+1]{R'} m_1$, which contradicts $j \xrightarrow[\ddot t+1]{R'} j_1 \in S'_{\ddot t+1}$. Thus, $m_1 \in S_{\ddot t+1}$. Suppose $m_1 \xrightarrow[\ddot t+1]{R} m_2$. Since $j \xrightarrow[\ddot t+1]{R} m_1$ and $k \in U_{\ddot t+1}$, then $m_2 \in U_{\ddot t+1}$ and $m_2 \preceq k$. Then, $m_2 \in U'_{\ddot t+1}$. Further, either $[m_2 \prec k]$ or $[m_2 = k$ and $m_1 \prec j_1]$. Since $j \not\xrightarrow[\ddot t+1]{R'} m_1$, we have $m_1 \not\xrightarrow[\ddot t+1]{R'} m_2$. Let $m_1 \xrightarrow[\ddot t+1]{R'} m'_2$. Since $m_2 \in U'_{\ddot t+1}$, we have $m'_2 \in U'_{\ddot t+1}$ and $m'_2 \prec m_2$. Then, $m'_2 \prec k$, which contradicts $j \xrightarrow[\ddot t+1]{R'} j_1$.



$\boldsymbol{h'_{\ddot{t}+1}(m_1)} \neq \boldsymbol{h_{\ddot{t}+1}(m_1)}$: Let $a \equiv h_{\ddot{t}+1}(m_1)$. By the *induction hypothesis*, since $h'_{\ddot{t}+1}(m_1) \neq a$, $m_1 \in CONN(i, R', \ddot{t})$. Thus, $m_1 \in CONN(i, R', \ddot{t}+1)$. Further, $m_1 \in S_{\ddot{t}+1}$. Since $O_{\ddot{t}+1} \subseteq O'_{\ddot{t}+1}$, there is $\hat{m} \in N'_{\ddot{t}+1}$ such that $h'_{\ddot{t}+1}(\hat{m}) = a$. Suppose $m_1 \xrightarrow[\ddot{t}+1]{R} m_2$. Since $j \xrightarrow[\ddot{t}+1]{R} m_1$, we have $m_2 \in U_{\ddot{t}+1} \subseteq U'_{\ddot{t}+1}$ and $m_2 \prec k$.

Since $j \not\xrightarrow[\ddot{t}+1]{R'} \hat{m}$, $\hat{m} \in S'_{\ddot{t}+1}$. Since $h_{\ddot{t}+1}(\hat{m}) \neq a$, by the *induction hypothesis*, $\hat{m} \in CONN(i, R', \ddot{t}+1)$. So there is a first $\hat{t}$ such that $\hat{m} \in CONN(i, R', \hat{t})$. Then, $h_{\hat{t}}(\hat{m}) = h'_{\hat{t}}(\hat{m}) = a$, and $\hat{m} \in S'_{\hat{t}} \subseteq S_{\hat{t}}$.

Now we consider the first $\check{t}$, which is between $\hat{t}$ and $\ddot{t}+1$, such that $h_{\check{t}}(m_1) = a$. Then, $m_1 \xrightarrow[\check{t}-1]{R} S_{\check{t}-1}$ which contradicts $m_2 \in U_{\ddot{t}+1} \subseteq U_{\check{t}-1}$.



Stage 5) Next we consider the people who point at satisfied people whose pointees satisfied and have unsatisfied pointees, under R'. Particularly, consider $j \in N'_{\ddot{t}+1} \setminus CONN(i, R', \ddot{t}+1)$ be such that $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R'} j_2 \in S'_{\ddot{t}+1} \xrightarrow[\ddot{t}+1]{R'} k \in U'_{\ddot{t}+1}$. Then, $j_1, j_2 \in S_{\ddot{t}+1}$.

By the preceding arguments, $j_1 \xrightarrow[\ddot{t}+1]{R} j_2 \xrightarrow[\ddot{t}+1]{R} k \in U_{\ddot{t}+1}$. Suppose $j \xrightarrow[\bar{t}]{R} m_1 \neq j_1$. Let $m_1 \xrightarrow[\ddot{t}+1]{R} m_2 \xrightarrow[\ddot{t}+1]{R} m_3$. We consider the following cases.

$h'_{\ddot{t}+1}(m_1) = h_{\ddot{t}+1}(m_1)$: If $m_1 \in U_{\ddot{t}+1}$, then $m_1 \in U'_{\ddot{t}+1}$ and $h_{\ddot{t}+1}(m_1) = h'_{\ddot{t}+1}(m_1) = \omega(m_1)$.

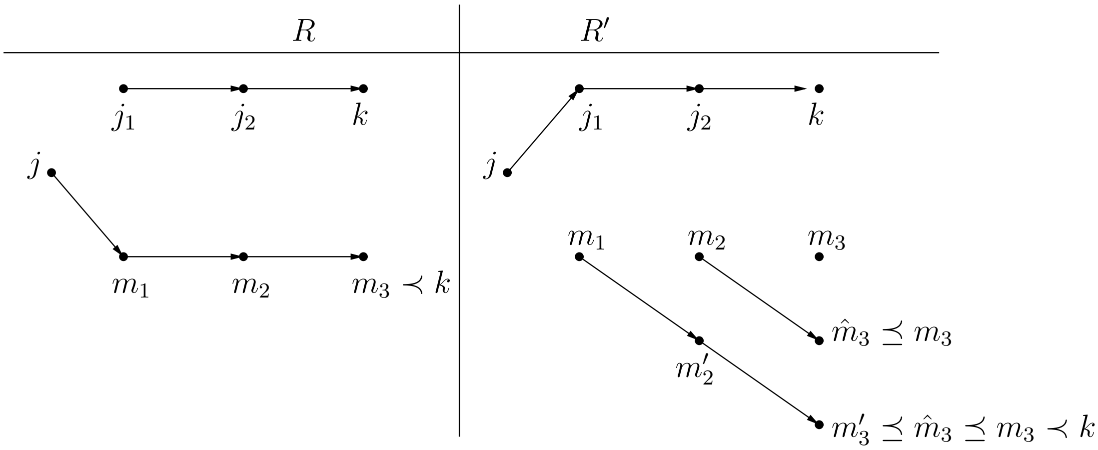
Then, $j \xrightarrow[\ddot{t}+1]{R'} m_1$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1}$. Thus, $m_1 \in S_{\ddot{t}+1}$. Two sub-cases are as follows:

$h'_{\ddot{t}+1}(m_2) = h_{\ddot{t}+1}(m_2)$: If $m_2 \in U_{\ddot{t}+1}$, then $m_2 \in U'_{\ddot{t}+1}$ and $h_{\ddot{t}+1}(m_2) = h'_{\ddot{t}+1}(m_2) = \omega(m_2)$. Then, $m_1 \xrightarrow[\ddot{t}+1]{R} U'_{\ddot{t}+1}$ and $j \xrightarrow[\ddot{t}+1]{R'} m_1$, which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1 \in S'_{\ddot{t}+1}$. Thus, $m_2 \in S_{\ddot{t}+1}$.

Since $j \xrightarrow[\ddot{t}+1]{R} m_1 \neq j_1$, $m_3 \in U_{\ddot{t}+1}$. Further, $m_3 \in U'_{\ddot{t}+1}$ and either $[m_3 \prec k]$ or $[m_3 = k$ and $m_1 \prec j_1]$. Since, $j \overset{R'}{\underset{\ddot{t}+1}{\not\longrightarrow}} m_1$, then either,

(a) $m_1 \xrightarrow[\ddot{t}+1]{R'} m_2 \xrightarrow[\ddot{t}+1]{R'} m'_3 \neq m_3$: Since $m_3 \in U'_{\ddot{t}+1}$, $m'_3 \in U'_{\ddot{t}+1}$ and $m'_3 \prec m_3 \prec k$. This contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1$.

(b) $m_1 \xrightarrow[\ddot{t}+1]{R'} m'_2 \neq m_2$: Since $j \xrightarrow[\ddot{t}+1]{R'} j_1 \neq m_1$, we have $m'_2 \in S'_{\ddot{t}+1}$. Suppose $m_2 \xrightarrow[\ddot{t}+1]{R'} \hat{m}_3$ and $m'_2 \xrightarrow[\ddot{t}+1]{R'} m'_3$. Since $m_3 \in U'_{\ddot{t}+1}$, $\hat{m}_3 \in U'_{\ddot{t}+1}$ and $\hat{m}_3 \preceq m_3$. Since $m'_2 \in S'_{\ddot{t}+1}$, $m_1 \xrightarrow[\ddot{t}+1]{R'} m'_2$, and $\hat{m}_3 \in U'_{\ddot{t}+1}$, we have $m'_3 \in U'_{\ddot{t}+1}$ and $m'_3 \preceq \hat{m}_3$. Thus, $m'_3 \prec k$ which contradicts $j \xrightarrow[\ddot{t}+1]{R'} j_1$.



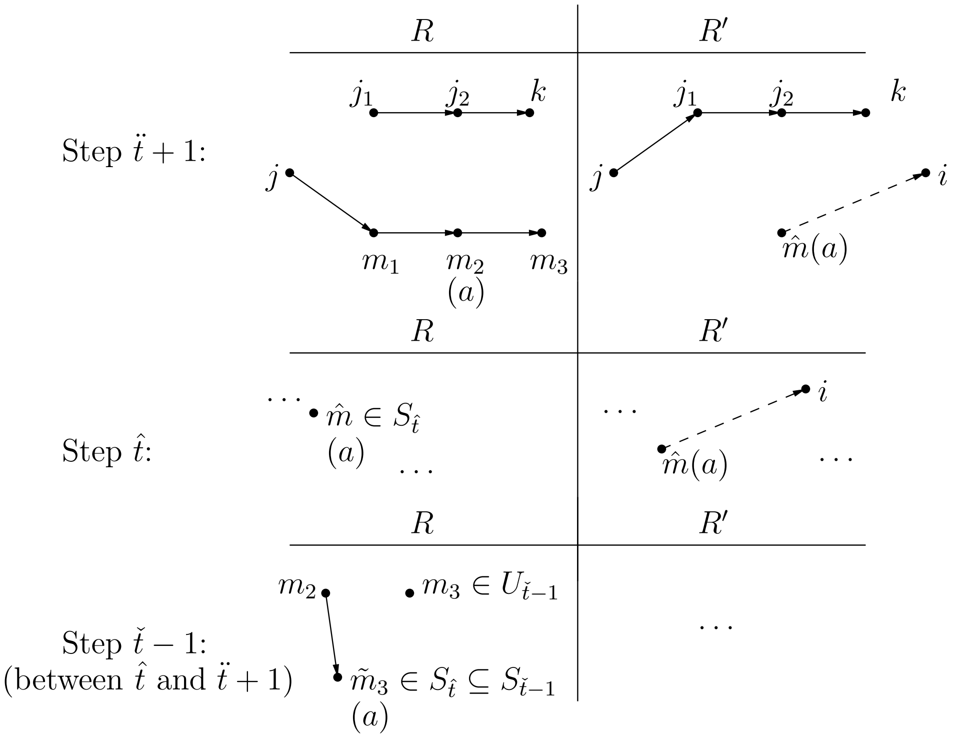
$\boldsymbol{h'_{\ddot t+1}(m_2) \neq h_{\ddot t+1}(m_2)}$: Let $a \equiv h_{\ddot t+1}(m_2)$. By the *induction hypothesis*, since $h'_{\ddot t+1}(m_2) \neq a$, we have $m_2 \in S_{\ddot t+1}$. Since $j \xrightarrow[\ddot t]{R} m_1$, $m_3 \in U_{\ddot t+1} \subseteq U'_{\ddot t+1}$.

Since $O_{\ddot t+1} \subseteq O'_{\ddot t+1}$, there is $\hat m \in N'_{\ddot t+1}$ such that $h'_{\ddot t+1}(\hat m) = a$ and by the *induction hypothesis*, $\hat m \in CONN(i, R', \ddot t + 1)$.

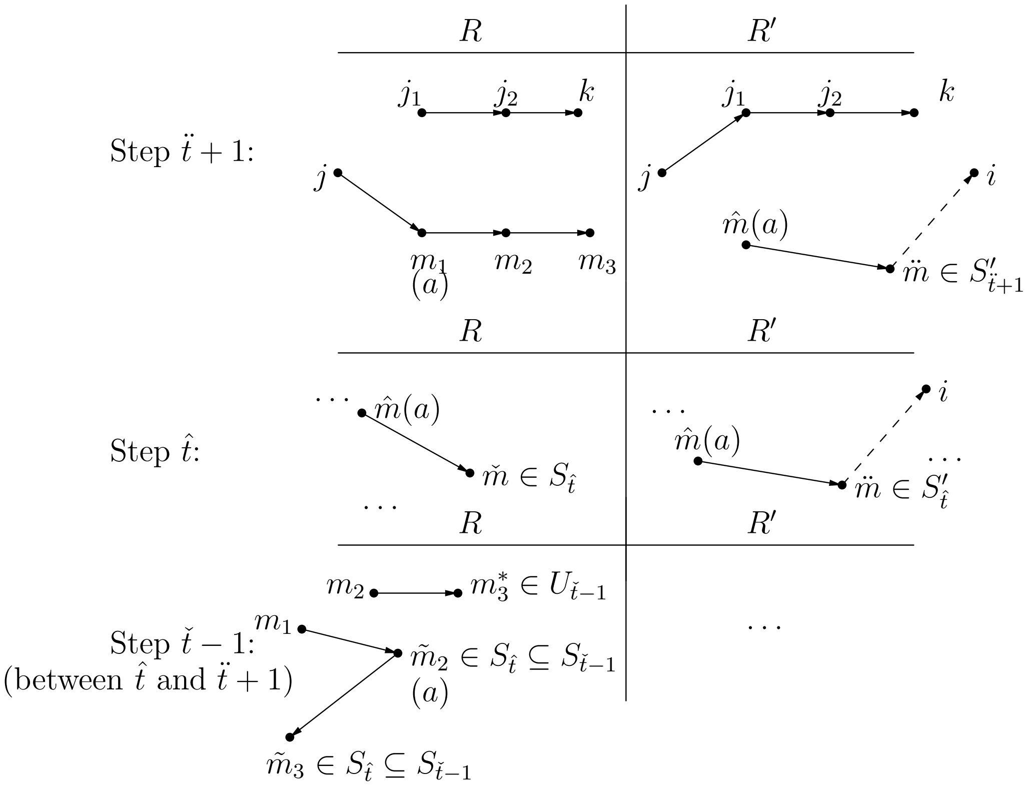
Since $a\ I_{m_1}\ h_{\ddot t+1}(m_1)$, and $j \not\xrightarrow[\ddot t+1]{R'} m_1$, we have that $m_1 \in S'_{\ddot t+1}$, $m_1 \xrightarrow[\ddot t+1]{R'} S'_{\ddot t+1}$, and $\hat m \in S'_{\ddot t+1}$.

Since $h_{\ddot t+1}(\hat m) \neq a$ and since there is a first $\hat t$ such that $\hat m \in CONN(i, R', \hat t)$, $h_{\hat t}(\hat m) = h'_{\hat t}(\hat m) = a$, and $\hat m \in S'_{\hat t} \subseteq S_{\hat t}$.

Now consider the first $\check t$, which is between $\hat t$ and $\ddot t + 1$, such that $h_{\check t}(m_2) = a$. Then, $m_2 \xrightarrow[\check t-1]{R} S_{\check t-1}$ which contradicts $m_3 \in U_{\ddot t+1} \subseteq U_{\check t-1}$.



$h'_{\ddot t+1}(m_1) \neq h_{\ddot t+1}(m_1)$: Let $a \equiv h_{\ddot t+1}(m_1)$. Since $h'_{\ddot t+1}(m_1) \neq a$, $m_1 \in S_{\ddot t+1}$. Since $O_{\ddot t+1} \subseteq O'_{\ddot t+1}$, there is $\hat m \in N'_{\ddot t+1}$ such that $h'_{\ddot t+1}(\hat m) = a$. Since $j \xrightarrow[\ddot t+1]{R'} j_1$, we have that $\hat m \in S'_{\ddot t+1}$ and $\hat m \xrightarrow[\ddot t+1]{R'} S'_{\ddot t+1}$. Since $h_{\ddot t+1}(\hat m) \neq a$, by the *induction hypothesis*, $\hat m \in CONN(i, R', \ddot t + 1)$ and there is a first $\hat t$ such that $\hat m \in CONN(i, R', \hat t)$. Since $\hat m \in S'_{\ddot t+1}$ and $p'_{\ddot t+1}(\hat m) = p'_{\hat t}(\hat m)$, we have that $\hat m \in S'_{\hat t}$. This implies that $\hat m \in S_{\hat t}$ and $h_{\hat t}(\hat m) = a$. Since $\hat m \xrightarrow[\hat t]{R'} S'_{\hat t}$, then $\hat m \xrightarrow[\hat t]{R} S_{\hat t}$. And for each $\hat{\hat t} > \hat t$, we have $\hat m \xrightarrow[\hat{\hat t}]{R} S_{\hat{\hat t}}$. Now consider the first $\check t$, which is between $\hat t$ and $\ddot t + 1$, such that $h_{\check t}(m_1) = a$. Then, $m_1 \xrightarrow[\check t-1]{R} S_{\check t-1} \xrightarrow[\check t-1]{R} S_{\check t-1}$. However, if $m_2 \in U_{\ddot t+1} \subseteq U_{\check t}$, then $m_1 \xrightarrow[\ddot t+1]{R} U_{\ddot t+1} \subseteq U_{\check t}$ and if $m_2 \in S_{\ddot t+1}$, then $m_3 \in U_{\ddot t+1}$ and $m_1 \xrightarrow[\ddot t+1]{R} S_{\ddot t+1} \xrightarrow[\ddot t+1]{R} U_{\ddot t+1}$. In either case, we have reached a contradiction.



Stage ...) Repeating this argument for the rest of the pointing phase we show (iii).

Now, we prove (v) for $\ddot{t}+1$. Suppose $j \in CONN(i,R,\ddot{t}+1) \setminus CONN(i,R',\ddot{t}+1)$. Then, there is $\{j_1, j_2, ..., j_r, i\} \subset N_{\ddot{t}+1} \subseteq N'_{\ddot{t}+1}$, such that $j \xrightarrow[\ddot{t}+1]{R} j_1 \xrightarrow[\ddot{t}+1]{R} j_2 \xrightarrow[\ddot{t}+1]{R} ... \xrightarrow[\ddot{t}+1]{R} j_r \xrightarrow[\ddot{t}+1]{R} i$. Since $j \notin CONN(i,R',\ddot{t}+1)$, by (iii), $j \xrightarrow[\ddot{t}+1]{R'} j_1$. Then, $j_1 \notin CONN(i,R',\ddot{t}+1)$. Again, by (iii), $j_1 \xrightarrow[\ddot{t}+1]{R'} j_2$ and $j_2 \notin CONN(i,R',\ddot{t}+1)$. Repeating the argument r times, $j_r \notin CONN(i,R',\ddot{t}+1)$. By (iii), $j_r \xrightarrow[\ddot{t}+1]{R'} i$, and this contradicts $j \notin CONN(i,R',\ddot{t}+1)$.

Finally, we prove (iv) for Step $\ddot{t}+1$. We show that for each $j \in N'_{\ddot{t}+1} \setminus CONN(i,R',\ddot{t}+1)$, $h_{\ddot{t}+1}(j) = h'_{\ddot{t}+1}(j)$. By (iii) each *trading cycle* that does not involve people connected to i under R' is also a *trading cycle* under R. Therefore, for each $j \in N'_{\ddot{t}+1} \setminus CONN(i,R',\ddot{t}+1)$, $h'_{\ddot{t}+2}(j) = h_{\ddot{t}+2}(j)$. Moreover, for each $j \in CONN(i,R',\ddot{t}+1)$, $h'_{\ddot{t}+2}(j) = h'_{\ddot{t}+1}(j)$. ◇

D List of abbreviations

Abbrev.	Description
O	Set of distinct objects.
N	Set of people.
$\omega : N \to O$	List of endowments.
$\omega(i)$	i's component of the endowment.
R_i	i's weak preference relation over O.
$\mathcal{R}$	Set of all preference relations over O.
R	Preference profile.
$\mathcal{R}^N$	Set of all preference profiles.
$\mathcal{P}$	Set of strict preference relations.
R_{-i}	Preference relation of everyone but i.
R_S	Preference profile of people in $S \subseteq N$.
R_{-S}	Preference profile of people not in $S \subseteq N$.
A	Set of allocations.
$\alpha(i)$	i's component of $\alpha \in A$.
$\alpha(S) = \bigcup_{i \in S}\{\alpha(i)\}$	Collective assignment to members of $S \in N$ under α.
$\varphi : \mathcal{R}^N \times A \to A$	Rule that selects an allocation for each problem.
$IR(R, \omega)$	Set of *individually rational* allocations.
$PE(R, \omega)$	Set of *efficient* allocations.
$C^W(R, \omega)$	Set of allocations in the weak core.
$C(R, \omega)$	Set of allocations in the core.
$\tau(R_i, O')$	Set of i's most preferred objects under R_i among O'.
$O_t \subseteq O$	Remaining objects determined after the *departure phase* in Step t of the *top cycles* algorithm.
$N_t \subseteq N$	Remaining people determined after the *departure phase* in Step t of the *top cycles* algorithm.
$h_{t+1} : N_t \to O_t$	Holding vector determined after the *trading phase* in Step t of the *top cycles* algorithm.
$p_t(i)$	Person whom i points at.
$i \underset{t}{\longrightarrow} j$	i points at j in Step t of the *top cycles* algorithm.
$i \underset{t}{\longrightarrow}\underset{t}{\longrightarrow} j$	i points at someone who is pointing at j in Step t of the *top cycles* algorithm.
$i \underset{t}{\longrightarrow} M$, $M \subset N_t$	i points at someone in $M \in N_t$ in Step t of the *top cycles* algorithm.
S_t	Set of satisfied people who hold one of their most preferred objects among O_t
$U_t = N_t \setminus S_t$	Set of unsatisfied people.

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