Stability and the Core of Probabilistic Marriage Problems

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Abstract

We study the marriage problem where a probability distribution over matchings is chosen. The "core" has been central to the the analysis of the deterministic problem. This is particularly the case since it coincides with the (conceptually) simpler set of "stable" matchings.

While the generalization of *stability* to the probabilistic stetting in an *ex post* sense is clear (Rothblum 1992), there are several natural *ex ante* generalizations: expected utility for given von Neumann-Morgenstern preferences, stochastic dominance requiring comparability, and stochastic dominance without requiring comparability.

The purpose of this paper is to work out the logical relations between the various concepts. Thus, we provide a foundation for further research on probabilistic matching.

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1 Introduction

The *core* of a marriage problem is the set of matchings where no group of people can be better off by re-matching among themselves. That is, the set of matchings that cannot be "blocked" by any group. This has been a central concept in the analysis of the deterministic matching problem (Gale and Shapley 1962). One of

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many nice features of these deterministic problems (Roth and Sotomayor 1990) is that the *core* property is equivalent to simply ruling out blocking pairs. That is, *stability* and the *core* property are equivalent.

As with other problems involving indivisibilities, randomization is one way to bring a sense of justice to the matching process (Aldershof, Carducci and Lorenc 1999, Bogomolnaia and Moulin 2004, Klaus and Klijn 2006, Sethuraman, Teo and Qian 2006). Furthermore, randomization in many cases allows us to make *ex ante Pareto improvements* over the set of *stable* deterministic allocations (Manjunath 2015).

Aside from these normative reasons for considering probabilistic matchings, such probabilistic matchings play a key role in the analysis of decentralized matching with search frictions by Lauermann and Nöldeke (2014). In such a setting, stability says that neither member of a matched pair "expects" to find a better mate upon rejecting his or her mate.

Stability of the matching that is eventually realized can be achieved by randomizing only among *stable* deterministic allocations (Sethuraman et al. 2006). We call this *ex post stability*. If (as in the model of Lauermann and Nöldeke (2014)) pairs are able to block a probability distribution itself (or say, prior to the realization of a matching), the analyst would need an *ex ante* notion of stability. If groups of people are able to commit, prior to realization, to probabilistic allocations among themselves, a notion of an *ex ante core* is called for. As we see in the sequel, the equivalence of *stability* and membership in the *core* no longer holds in this context. Nonetheless, both are worth considering based on the application.

There are several natural ways to define *ex ante* blocking depending on how we compare lotteries. If the primitives of the model include cardinal information, then *expected utility* is the natural choice: no member of the blocking group receives a lottery that yields a lower expected utility than the original lottery. Going only by ordinal preferences, lotteries can be compared using stochastic dominance. For some person i, let π_i and π'_i be two distinct lotteries over i's potential mates. Suppose that for each potential mate j, i is at least as likely to prefer his mate to j under π_i as π'_i. Then we say that π_i stochastically dominates π'_i at i's preference relation. Of course, this only provides an incomplete ordering of lotteries for each person. There are, therefore, two ways of proceeding when it comes to defining a blocking coalition. The first is very demanding: after re-matching among themselves, each member of the group enjoys a lottery that stochastically dominates the original one. The second is more permissive: re-matching among the group does not make any of its members worse off in a stochastic dominance sense.

Corresponding to each definition of *ex ante blocking*, we define versions *stability* and the *core*.

Our contribution in this paper is mainly conceptual. We formalize various *core* and *stability* notions and analyze their logical relations. While several of the concepts presented here have appeared in the literature, either formally stated or informally, we are not aware of any other systematic approach to this topic.

The remainder of the paper is organized as follows: In Section 2, we present the model and in Section 3 we provide formal definitions of the *several blocking* notions and the related *stability* and *core* concepts. In section 4, we present the logical relationships between these concepts. In Section 5, we show for all but one of these concepts the existence of a matching that satisfies its conditions. We conclude in Section 6.

2 The Model

Let M be a set of men and W a set of women. Let $N \equiv M \cup W$ be the set of all people. For each man $m \in M$, his set of potential mates is $S_m \equiv W \cup \{m\}$. Similarly, for each $w \in W$, her set of potential mates is $S_w \equiv M \cup \{w\}$. Each $m \in M$ has a preference relation[1] over S_m and each woman $w \in W$ has a preference relation over S_w. A **problem** consists of a preference relation for each person: $R \equiv (R_M, R_W) \in \mathcal{R}^N$. A **deterministic matching**, $\mu : N \to N$, either assigns each person a mate or leaves him/her single. That is, for each $i \in N$,

$$\begin{array}{ll} i) & \mu(i) \in S_i \text{ and} \\ ii) & \mu(\mu(i)) = i. \end{array}$$

Let $\mathbb{M}$ be the set of all *deterministic matchings.*

A **probabilistic matching** is a probability distribution π over $\mathbb{M}$. Let $\Pi \equiv \Delta(\mathbb{M})$ be the set of all probabilistic matchings. For each $\pi \in \Pi$ and each $i \in N$, π_i is π's marginal distribution over i's mate: for each $j \in S_i$, $\pi_i(j) \equiv \pi(\{\mu \in \mathbb{M} : \mu(i) = j\})$.

For each problem $R \in \mathcal{R}^N$, let $\mathcal{U}(R)$ be the set of all *von Neumann-Morgenstern* indices that are **consistent with** R. That is,

$$\mathcal{U}(R) \equiv \left\{ u \in \mathbb{R}_+^{|M|(|W|+1)+|W|(|M|+1)} : \begin{array}{l} \text{for each } i \in N \\ \text{and each pair } j,k \in S_i, \end{array} u_i(j) > u_i(k) \Leftrightarrow j \; P_i \; k \right\}.$$

For each $u \in \mathcal{U}(R)$, let $U_i : \Delta(S_i) \to \mathbb{R}_+$ be defined as follows: For each $\pi_i \in \Delta(S_i)$,

$$U_i(\pi_i) \equiv \sum_{j \in S_i} \pi_i(j) \cdot u_i(j).$$

[1]Preference relations are linear orders.

For each $i \in N$, each $R \in \mathcal{R}^N$, and each pair $\pi_i, \pi'_i \in \Pi$, $\boldsymbol{\pi_i}$ **stochastically dominates** $\boldsymbol{\pi'_i}$ **at** $\boldsymbol{R_i}$, **(**$\boldsymbol{\pi_i \; P_i^{sd} \; \pi'_i}$**)** if,

$i)$ $j_1 \; P_i \; j_2 \; P_i \ldots$ and

$$ii) \quad \begin{array}{r} \pi_i(j_1) \geq \pi'_i(j_1), \\ \pi_i(j_2) + \pi_i(j_1) \geq \pi'_i(j_1) + \pi'_i(j_2), \\ \pi_i(j_3) + \pi_i(j_2) + \pi_i(j_1) \geq \pi'_i(j_1) + \pi'_i(j_2) + \pi'_i(j_3), \\ \vdots \end{array}$$

3 Stability and core notions

The most natural notion of blocking a lottery is that a group of people might walk away from the lottery's realization if they can do better among themselves.

For each $R \in \mathcal{R}^N$ and each $\mu \in \mathbb{M}$, a group $S \subseteq N$ **blocks** $\boldsymbol{\mu}$ **at** $\boldsymbol{R}$ if there is $\mu' \in \mathbb{M}$ such that $\mu'(S) = S$ and for each $i \in S, \mu'_i \; P_i \; \mu_i$.[2] If there is no $S \subseteq N$ that *blocks* μ *at* R, then μ is a **stable matching at** $\boldsymbol{R}$. Let $\mathcal{S}(R) \subseteq \mathbb{M}$ be the set of all *stable matchings*.

For each $R \in \mathcal{R}^N$ and each $\pi \in \Pi$, a group $S \subseteq N$ **blocks** $\boldsymbol{\pi}$ **ex post at** $\boldsymbol{R}$ if there is $\mu \in \text{supp}(\pi)$ and $\mu' \in \mathbb{M}$ such that for each $i \in S$,

$i)$ $\mu'(i) \in S$, and
$ii)$ $\mu'(i) \; P_i \; \mu(i)$.

If there is no $S \subseteq N$ that *blocks* π *ex post at* R, then π is an **ex post core matching at** $\boldsymbol{R}$. Let $\boldsymbol{\mathcal{C}^{xp}(R)} \subseteq \Pi$ be the set of all *ex post core matchings*.

If there is no $S \subseteq N$ such that $|S| \leq 2$ and S *blocks* π *ex post*, then π is an **ex post stable matching at** $\boldsymbol{R}$. Let $\boldsymbol{\mathcal{S}^{xp}(R)} \subseteq \Pi$ be the set of all *ex post stable matchings*.

As we have explained earlier, when people can commit to running a "private" lottery amongst themselves, we need to think about *ex ante* blocking as well. The next three blocking notions are based on this idea. Members of a blocking coalition need to compare the lottery under consideration with those that they can come up with on their own. The first way to do this is using *expected utility*.

For each $u \in \mathcal{U}(R)$, we say that $S \subseteq N$ **expected-utility-blocks** $\boldsymbol{\pi}$ **at** $\boldsymbol{u}$ if there is $\pi' \in \Pi$ such that for each $i \in S$,

$i)$ $\pi'_i(S) = 1$, and
$ii)$ $U_i(\pi'_i) > U(\pi_i)$.

[2]It is easy to show that restricting our attention to coalitions of no more than two people does not make a difference. See Roth and Sotomayor (1990).

If there is no $S \subseteq N$ that *expected-utility-blocks π at u*, then π is an **expected utility core matching at u**. Let $\boldsymbol{\mathcal{C}^{eu}(u)} \subseteq \Pi$ be the set of all *expected utility core matchings at u*.

If there is no $S \subseteq N$ such that $|S| \leq 2$ and S *expected-utility-blocks π at u*, then π is an **expected utility stable matching at u**. Let $\boldsymbol{\mathcal{S}^{eu}(u)} \subseteq \Pi$ be the set of all *expected utility stable matchings at u*.

Without cardinal information, we rely on stochastic dominance to compare lotteries: $S \subseteq N$ **strongly stochastic-dominance-blocks π at R** if there is $\pi' \in \Pi$ such that for each $i \in S$,

$$\begin{array}{ll} i) & \pi'_i(S) = 1, \text{ and} \\ ii) & \pi'_i \; P^{sd}_i \; \pi_i. \end{array}$$[3]

This requires that each $i \in S$ is better off at π' than at π.[4] If there is no $S \subseteq N$ that *strongly stochastic-dominance-blocks π at R*, then π is a **weak stochastic dominance core matching at R**. Let $\boldsymbol{\mathcal{C}^{wsd}(R)} \subseteq \Pi$ be the set of all *weak stochastic dominance core matchings at R*.

If there is no $S \subseteq N$ such that $|S| \leq 2$ and S *strongly stochastic-dominance-blocks π at R*, then π is a **weakly stochastic dominance stable matching at R**. Let $\boldsymbol{\mathcal{S}^{wsd}(R)} \subseteq \Pi$ be the set of all *weakly stochastic dominance stable matchings at R*.

We can be less demanding of blocking coalitions: $S \subseteq N$ **weakly stochastic-dominance-blocks π at R** if there is $\pi' \in \Pi$ such that for each $i \in S, \pi'_i(S) = 1$ and there is no $i \in S$ such that $\pi_i \neq \pi'_i \; \pi_i \; P^{sd}_i \; \pi'_i$.

If there is no $S \subseteq N$ that *weakly stochastic-dominance-blocks π at R*, then π is a **strong stochastic dominance core matching at R**. Let $\boldsymbol{\mathcal{C}^{ssd}(R)} \subseteq \Pi$ be the set of all *strong stochastic dominance core matchings at R*.

If there is no $S \subseteq N$ such that $|S| \leq 2$ and S *weakly stochastic-dominance-blocks π at R*, then π is a **strong stochastic dominance stable matching at R**. Let $\boldsymbol{\mathcal{S}^{sd}(R)} \subseteq \Pi$ be the set of all *strong stochastic dominance stable matchings at R*.

4 Logical relations

We start with an overview of the relationship between *stability* notions:

$$\mathcal{S}(R) \subseteq \mathcal{S}^{ssd}(R) \subseteq \begin{array}{c} \mathcal{S}^{xp}(R) \\ \rotatebox{90}{$\not\subseteq$}\rotatebox{90}{$\not\supseteq$} \\ \mathcal{S}^{eu}(u) \end{array} \subseteq \mathcal{S}^{wsd}(R).$$

[3]This is the notion of dominance that Bogomolnaia and Moulin (2001) use in their definition of "ordinal efficiency" for the probabilistic assignment problem.

[4]That is, for every $u \in \mathcal{U}(R), U_i(\pi') > U_i(\pi)$.

The relationship between the different *cores* mirrors the relationships between *stability* notions:

$$\mathcal{C}^{ssd}(R) \subseteq \begin{array}{c} \mathcal{C}^{xp}(R) \\ \not\subseteq \not\supseteq \\ \mathcal{C}^{eu}(u) \end{array} \subseteq \mathcal{C}^{wsd}(R).$$

The relationships across the two groups are not summarized as simply, so we defer this until the end of the section. It is clear, however, from the definitions that each *stability* notion is implied by the corresponding *core* notion. That is, for each $R \in \mathcal{R}^N$ and each $u \in \mathcal{U}(R)$,

$$\begin{array}{lcl} \mathcal{C}^{xp}(R) & \subseteq & \mathcal{S}^{xp}(R), \\ \mathcal{C}^{eu}(u) & \subseteq & \mathcal{S}^{eu}(u), \\ \mathcal{C}^{wsd}(R) & \subseteq & \mathcal{S}^{wsd}(R), \text{ and} \\ \mathcal{C}^{ssd}(R) & \subseteq & \mathcal{S}^{ssd}(R). \end{array}$$

It is easy to show that $\mathcal{S}^{xp}(R) \supseteq \mathcal{C}^{xp}(R)$ (Roth and Sotomayor 1990). From the definitions, it is also easy to show that

$$\mathcal{S}^{wsd}(R) = \bigcup_{u \in \mathcal{U}(R)} \mathcal{S}^{eu}(u), \qquad \mathcal{C}^{wsd}(R) = \bigcup_{u \in \mathcal{U}(R)} \mathcal{C}^{eu}(u),$$

$$\mathcal{S}^{ssd}(R) = \bigcap_{u \in \mathcal{U}(R)} \mathcal{S}^{eu}(u), \text{ and } \mathcal{C}^{ssd}(R) = \bigcap_{u \in \mathcal{U}(R)} \mathcal{C}^{eu}(u).$$

Thus, $\mathcal{S}^{ssd}(R) \subseteq \mathcal{S}^{wsd}(R)$ and $\mathcal{C}^{ssd}(R) \subseteq \mathcal{C}^{wsd}(R)$.

To start with, we re-state the following result relating stable deterministic matchings to ex post stable matchings by Sethuraman et al. (2006).

Proposition 1. *For each $R \in \mathcal{R}$, $S^{xp}(R)$ is the convex hull of $S(R)$.*

Example 1. *There are $R \in \mathcal{R}^N$ and $u \in \mathcal{U}(R)$ such that $\mathcal{S}^{xp}(R) \not\subseteq \mathcal{S}^{eu}(u)$*

Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$ and suppose preferences are as follows:

m_1	m_2	m_3	w_1	w_2	w_3
w_1	w_2	w_3	m_2	m_3	m_1
w_2	w_3	w_1	m_3	m_1	m_2
w_3	w_1	w_2	m_1	m_2	m_3
m_1	m_2	m_3	w_1	w_2	w_3

Let $u \in \mathcal{U}(R)$ be such that for each pair $i, j \in N$, if $j \in S_i$ is i's first choice, $u_i(j) = 1$, if j is i's second choice $u_i(j) = 0.99$, if j is i's third choice $u_i(j) = 0.01$

and $u_i(i) = 0$. Let μ^1, μ^2, and $\mu^3 \in \mathbb{M}$ be as follows:

μ^1	μ^2	μ^3
$m_1 \leftrightarrow w_1$	$m_1 \leftrightarrow w_2$	$m_1 \leftrightarrow w_3$
$m_2 \leftrightarrow w_2$	$m_2 \leftrightarrow w_3$	$m_2 \leftrightarrow w_1$
$m_3 \leftrightarrow w_3$	$m_2 \leftrightarrow w_1$	$m_3 \leftrightarrow w_2$

Let $\pi \in \Pi$ be such that $\pi(\mu^1) = \pi(\mu^2) = \pi(\mu^3) = \frac{1}{3}$. It is easy to check that $\pi \in \mathcal{S}^{xp}(R)$. For each $i \in N$, $U_i(\pi_i) = \frac{1}{3}1 + \frac{1}{3}0.99 + \frac{1}{3}0.01 = 0.6666$. Though, $u_{m_1}(w_2) = u_{w_2}(m_1) = 0.99 > 0.6666 = U_{m_1}(\pi_{m_1}) = U_{w_3}(\pi_{w_3})$. Thus, $\{m_1, w_2\}$ *expected-utility-block* π *at* u and $\pi \notin \mathcal{S}^{eu}(u)$. $\circ$

From Example 1, we also deduce that $\mathcal{S}^{xp}(R) \not\subseteq \mathcal{S}^{ssd}(R) = \bigcap_{u \in \mathcal{U}(R)} \mathcal{S}^{eu}(u)$.

In the sequel, for each pair $i, j \in N$, let $\delta^{ij} \in \Pi$ be such that $\delta^{ij}_i(j) = \delta^{ij}_j(i) = 1$ and for all $k \in N \setminus \{i, j\}$, $\delta^{ij}_k(k) = 1$.

Proposition 2. *For each $R \in \mathcal{R}^N$, $\mathcal{S}^{ssd}(R) \subseteq \mathcal{S}^{xp}(R)$*

Proof: Suppose $\pi \notin \mathcal{S}^{xp}(R)$. Then, there is $\mu \in$supp(π) such that one of the following cases applies:

Case 1: There is $i \in N$ such that $i \; P_i \; \mu_i$. Then, it cannot be that $\pi_i P^{sd}_i \delta^{ii}_i$. Thus, $\pi \notin \mathcal{S}^{ssd}(R)$.

Case 2: There are $i \in N$ and $j \in S_i \setminus \{i\}$ such that $j \; P_i \; \mu_i$ and $i \; P_j \; \mu_j$. Then neither $\pi_i \; P^{sd}_i \; \delta^{ij}_i$ nor $\pi_j \; P^{sd}_j \; \delta^{ij}_j$. Thus, $\pi \notin \mathcal{S}^{xp}(R)$. $\square$

Since $C^{ssd} \subseteq C^{xp}$ and $C^{xp} = S^{xp}$, Proposition 2 implies that $C^{ssd} \subseteq C^{xp}$.

Example 2. *There are $R \in \mathcal{R}^N$ and $u \in \mathcal{U}(R)$ such that $\mathcal{C}^{eu}(u) \not\subseteq \mathcal{S}^{xp}(R)$*

Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$ and suppose preferences are as follows:

m_1	m_2	w_1	w_2
w_1	w_2	m_2	m_1
m_1	m_2	w_1	w_2
w_2	w_1	m_1	m_2

It is easy to see that $\mathcal{S}^{xp}(R) = \{\delta^{m_1 m_1}\}$.

Let $u \in \mathcal{U}(R)$ be such that for each i, if $j \in S_i$ is i's first choice, $u_i(j) = 1$, if j is i's third choice $u_i(j) = 0$, and $u_i(i) = 0.01$. Let μ^1 and $\mu^2 \in \mathbb{M}$ be as follows:

μ^1	μ^2
$m_1 \leftrightarrow w_1$	$m_1 \leftrightarrow w_2$
$m_2 \leftrightarrow w_2$	$m_2 \leftrightarrow w_1$

Let $\pi \in \Pi$ be such that $\pi(\mu^1) = \pi(\mu^2) = \frac{1}{2}$. For each $i \in N, U_i(\pi_i) = 0.5$. We will show that $\pi \in \mathcal{C}^{eu}(u)$ even though $\pi \notin \mathcal{S}^{xp}(R)$. Suppose not. Then, there is $S \subseteq N$ such that S *expected-utility-blocks* π *at* u.

Case 1: $S = \{i\}$. However, $U_i(\pi_i) = 0.5 > 0.01 = U_i(\delta_i^{ii})$.

Case 2: $S = \{i, j\}$. Then, $j \in S_i \setminus \{i\}$. But $U_i(\delta_i^{ij}) = 0 < 0.5$ and $U_j(\delta_j^{ij}) = 0 < 0.5$.

Case 3: $S = \{i, j, k\}$. Then $\{j, k\} = S_i \setminus \{i\}$. Suppose S *expected-utility-blocks* π *at* u via π'. Then, either $U_j(\pi'_j) \leq 0.01 < 0.5$ or $U_k(\pi'_j) \leq 0.01 < 0.5$.

Case 4: $S = N$. Suppose N blocks π via π'. If $U_{m_1}(\pi'_{m_1}) > U_{m_1}(\pi_{m_1})$, then $\pi'_{w_1}(m_1) = \pi'_{m_1}(w_1) > \pi_{m_1}(w_1) = \frac{1}{2}$. So, $\pi'_{w_1}(\{m_2, w_1\}) < \frac{1}{2}$. Thus, $U_{w_1}(\pi'_{w_1}) < 0.5 = U_{w_1}(\pi)$. The analysis is symmetric for each other person in N.

In each case, we have shown that S cannot *expected-utility-block* π *at* u and thus, $\pi \in \mathcal{C}^{eu}(u)$. ◦

Example 2 establishes that $\mathcal{S}^{xp}(R) \not\supseteq \mathcal{C}^{wsd}(R) = \bigcap_{u \in \mathcal{U}(R)} \mathcal{C}^{eu}(u)$.

Proposition 3. *For each* $R \in \mathcal{R}^N$, $\mathcal{S}^{xp}(R) \subseteq \mathcal{C}^{wsd}(R)$.

Proof: Let $\pi \in \mathcal{S}^{xp}(R)$. The following inequality holds for each $m \in M$ and $w \in W$ (Rothblum 1992):

$$\sum_{w' R_m w} \pi_m(w') + \sum_{m' P_w m} \pi_w(m') \geq 1.$$

For the sake of contradiction, suppose that $\pi \notin \mathcal{C}^{wsd}(R)$. Then, there are $S \subseteq N$ and $\pi^S \in \Pi$ such that S *strongly stochastic-dominance-blocks* π *at* R through $\pi^S \in \Pi$. Then, for each $m \in S \cap M$ and $w \in S \cap W$,

$$\sum_{w' R_m w} \pi^S_m(w') + \sum_{m' P_w m} \pi^S_w(m') \geq \sum_{w' R_m w} \pi_m(w') + \sum_{m' P_w m} \pi_w(m') \geq 1.$$

Also, there are $m \in S \cap M$ and $w \in S \cap W$,

$$\pi^S_m(w) > 0 \text{ and } \sum_{w' R_m w} \pi^S_m(w') + \sum_{m' P_w m} \pi^S_w(m') > 1.$$

However, this implies that there are $m' \in S \cap M \setminus \{m\}$ and $w' \in S \cap W \setminus \{w\}$ such that the following inequality holds (Lemma 9 of Roth, Rothblum and Vande Vate (1993)):

$$\sum_{w'' R_{m'} w'} \pi^S_{m'}(w'') + \sum_{m'' P_{w'} m'} \pi^S_{w'}(m'') < 1.$$

Since $\sum_{w'' R_{m'} w'} \pi_{m'}(w'') + \sum_{m'' P_{w'} m'} \pi_{w'}(m'') \geq 1$, either

$$\begin{array}{ll} i) & \sum_{w'' R_{m'} w'} \pi^S_{m'}(w'') < \sum_{w'' R_{m'} w'} \pi_{m'}(w'') \text{ or} \\ ii) & \sum_{m'' P_{w'} m'} \pi^S_{w'}(m'') < \sum_{m'' P_{w'} m'} \pi_{w'}(m''). \end{array}$$

In either case, we have contradicted the assumption that for each $i \in S$, $\pi^S_i \; P^{sd}_i \; \pi_i$. Thus, $\mathcal{S}^{xp}(R) \subseteq \mathcal{C}^{wsd}(R)$. □

From Propositions 2 and 3, Examples 1 and 2, and the definitions of $\mathcal{S}^{ssd}$ and $\mathcal{S}^{wsd}$, for each $R \in \mathcal{R}^N$ and each $u \in \mathcal{U}(R)$,

$$\mathcal{S}^{ssd}(R) \subseteq \begin{array}{c} \mathcal{S}^{xp}(R) \\ \not\subseteq \not\supseteq \\ \mathcal{S}^{eu}(u) \end{array} \subseteq \mathcal{S}^{wsd}(R).$$

Example 3. *There are $R \in \mathcal{R}^N$ and $u \in \mathcal{U}(R)$ such that $\mathcal{S}^{ssd}(R) \not\subseteq \mathcal{C}^{eu}(u)$.*

Let $N \equiv \{m_1, m_2, m_3, w_1, w_2, w_3\}$, and let $R \in \mathcal{R}^N$ and $u \in \mathcal{U}(R)$ be as follows:

u_{m_1}	R_{m_1}	u_{m_2}	R_{m_2}	u_{m_3}	R_{m_3}	u_{w_1}	R_{w_1}	u_{w_2}	R_{w_2}	u_{w_3}	R_{w_3}
1	w_1	1	w_2	1	w_1	0.5	m_2	0.5	m_1	1	m_1
0.5	w_3	0.5	w_3	0.5	w_2	0.05	m_3	0.05	m_3	0.5	m_2
0.05	w_2	0.05	w_1	0.05	w_3	0.025	m_1	0.025	m_2	0.05	m_3
0	m_1	0	m_2	0	m_3	0	w_1	0	w_2	0	w_3

Let the pair $\mu^1, \mu^2 \in \mathbb{M}$ be as follows:

μ^1	μ^2
$m_1 \leftrightarrow w_3$	$m_1 \leftrightarrow w_3$
$m_2 \leftrightarrow w_1$	$m_2 \leftrightarrow w_2$
$m_3 \leftrightarrow w_2$	$m_3 \leftrightarrow w_1$

Let $\pi \in \Pi$ be such that $\pi(\mu^1) = \pi(\mu^2) = \frac{1}{2}$.

Let $\mu^{1'}, \mu^{2'} \in \mathbb{M}$ be as follows:

$\mu^{1'}$	$\mu^{2'}$
$m_1 \leftrightarrow w_1$	$m_1 \leftrightarrow w_2$
$m_2 \leftrightarrow w_2$	$m_2 \leftrightarrow w_1$
$m_3 \leftrightarrow w_3$	$m_3 \leftrightarrow w_3$

Let $\pi' \in \Pi$ be such that $\pi(\mu^{1'}) = \frac{1}{4}$ and $\pi(\mu^{2'}) = \frac{3}{4}$.

First, we show that $\pi \in \mathcal{S}^{ssd}(R)$. Note that for each $i \in N, u_i(i) = 0 < u_i(\pi_i)$. Thus, no individual *weakly stochastic-dominance-blocks* π *at* R. If m_1 is part of a blocking pair, it is with w_1. However, $\pi_{w_1}\ P^{sd}_{w_1}\ \delta^{m_1 w_1}_{w_1}$ and so $\{m_1, w_1\}$ is not a blocking pair. If m_2 is part of a blocking pair, it is with w_2. However, $\pi_{w_2}\ P^{sd}_{w_2}\ \delta^{m_2 w_2}_{w_2}$ and so $\{m_2, w_2\}$ is not a blocking pair. Finally, if m_3 is part of a blocking pair, it is with w_1. Again, $\pi_{w_1}\ P^{sd}_{w_1}\ \delta^{m_3 w_1}_{w_1}$ and so $\{m_3, w_1\}$ is not a blocking pair. Thus, $\pi \in \mathcal{S}^{ssd}(R)$.

Next, we show that $\pi \notin \mathcal{C}^{eu}(R)$. To see this, we compute the expected utilities of each member of $S \equiv \{m_1, m_2, w_1, w_2\}$ at π and π':

	π	π'
U_{m_1}	0.5	0.625
U_{m_2}	0.275	0.325
U_{w_1}	0.275	0.38125
U_{w_2}	0.0375	0.38125

Since for each $i \in S$, $U_i(\pi'_i) > U_i(\pi_i)$, $\pi \notin \mathcal{C}^{eu}(u)$. ◦

Example 3 also establishes that $\mathcal{C}^{ssd}(R) \subsetneq \mathcal{S}^{ssd}(R)$ and $\mathcal{S}^{eu}(u) \subsetneq \mathcal{C}^{eu}(u)$.

Example 4. *There is* $R \in \mathcal{R}^N$ *such that* $\mathcal{C}^{wsd}(R) \subsetneq \mathcal{S}^{wsd}(R)$.

Let $N \equiv \{m_1, m_2, m_3, w_1, w_2, w_3\}$. Let $R \in \mathcal{R}^N$ and the pair $\pi, \pi' \in \Pi$ be as follows:

π_{m_1}	R_{m_1}	π'_{m_1}	π_{m_2}	R_{m_2}	π'_{m_2}	π_{m_3}	R_{m_3}	π'_{m_3}
0.50	w_1	0.50	0.50	w_2	0.50	0.25	w_1	0
0.25	w_2	0.50	0.25	w_1	0.50	0.25	w_2	0.00
0.25	w_3	0.00	0.25	w_3	0.00	0.50	w_3	1.00
0.00	m_1	0.00	0.00	m_2	0.00	0.00	m_3	0.00

π_{w_1}	R_{w_1}	π'_{w_1}	π_{w_2}	R_{w_2}	π'_{w_2}	π_{w_3}	R_{w_3}	π'_{w_3}
0.25	m_2	0.50	0.25	m_1	0.50	0.25	m_3	0.00
0.50	m_1	0.50	0.50	m_2	0.50	0.25	m_2	0.00
0.25	m_3	0.00	0.25	m_3	0.00	0.50	m_3	1.00
0.00	w_1	0.00	0.00	w_2	0.00	0.00	w_3	0.00

Let $S \equiv \{m_1, m_2, w_1, w_2\}$. Clearly, for each $i \in S, \pi'_i\ P^{sd}_i\ \pi_i$. Thus, $\pi \notin \mathcal{C}^{wsd}(R)$. However, we show that $\pi \in \mathcal{S}^{wsd}(R)$. Since for each $i \in N$ and $j \in S_i, j\ R_i\ i$, no individual *strongly stochastic-dominance-blocks* π *at* R. If m_1 is in a blocking pair, it is with w_1. However, it is not the case that $\delta^{m_1 w_1}_{w_1}\ P^{sd}_{w_1}\ \pi_{w_1}$. If m_2 is in a blocking pair, it is with w_2. Again, it is not the case that $\delta^{m_2 w_2}_{w_2}\ P^{sd}_{w_2}\ \pi_{w_2}$. Finally, m_3 can only block with w_1 and it is not the case that $\delta^{m_3 w_1}_{w_1}\ P^{sd}_{w_1}\ \pi_{w_1}$. ◦

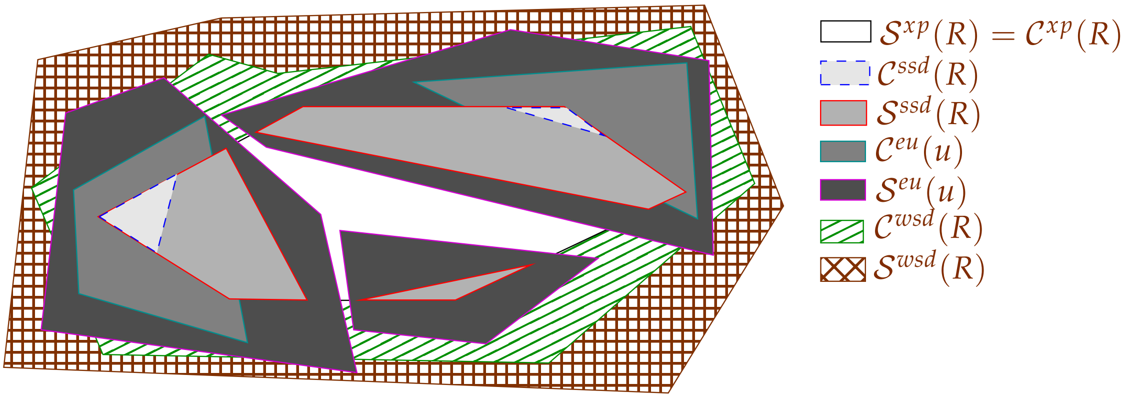

Figure 1: A schematic representation of the inclusion relations between the various sets.

In summary (see Figure 1), we have the following inclusion relations: For each $R \in \mathcal{R}^N$, and $u \in \mathcal{U}(R)$,

$$\begin{array}{ccccccc}
 & & & & \mathcal{C}^{xp}(R) & & \\
 & & & & \shortparallel & & \\
 & & & & \mathcal{S}^{xp}(R) & \subseteq & \mathcal{C}^{wsd}(R) \\
 & & & \rotatebox{45}{$\subseteq$} & & & \cap| \\
S(R) & \subseteq & \mathcal{S}^{ssd}(R) & & & & \mathcal{S}^{wsd}(R) \\
 & & & \rotatebox{-45}{$\supseteq$} & & \rotatebox{45}{$\subseteq$} & \\
 & & \cup| & & \mathcal{S}^{eu}(u) & & \\
 & & & & \cup| & & \\
 & & \mathcal{C}^{ssd}(R) & \subseteq & \mathcal{C}^{eu}(u) & &
\end{array}
\quad \text{and } \mathcal{C}^{eu}(u) \subseteq \mathcal{C}^{wsd}(R).$$

All of the above inclusion relations commute. Each pair of sets that are not shown as related have no inclusion relation.

5 Existence (or lack of it)

We show that a stable matching of each variety, with exception of the *strongly stochastic dominance stable matchings*, exists. We begin by stating that a *stable matchings* always exists (Gale and Shapley 1962).

Proposition 4. *For each $R \in \mathcal{R}^N$, $S(R) \neq \varnothing$.*

From the inclusion relations in Section 4, we have the following proposition.

Proposition 5. *For each $R \in \mathcal{R}^N$, and $u \in \mathcal{U}(R)$,*

$$\left.\begin{array}{l} \mathcal{S}^{ssd}(R) \\ \mathcal{S}^{xp}(R) \\ \mathcal{C}^{xp}(R) \\ \mathcal{S}^{wsd}(R) \\ \mathcal{C}^{wsd}(R) \\ \mathcal{S}^{xp}(u) \end{array}\right\} \neq \varnothing.$$

The following example proves that for some profiles of preferences, there may be no *strongly stochastic dominance stable matching.*

Example 5. *Emptiness of $\mathcal{C}^{ssd}$.*

Let $N = \{m_1, m_2, m_3, w_1, w_2, w_3\}$ and let $R \in \mathcal{R}^N$ be as follows:

m_1	m_2	m_3	w_1	w_2	w_3
w_1	w_2	w_3	m_2	m_3	m_1
m_1	m_2	m_3	w_1	w_2	w_3
w_2	w_3	w_1	m_3	m_1	m_2
w_3	w_1	w_3	m_1	m_2	m_3

Suppose that $\pi \in \mathcal{C}^{ssd}(R)$. Then, we show that $\pi_i(i) = 1$. If not, there is a pair $k, l \in \{1,2,3\}$ such that $\pi_{m_l}(w_k) > 0$. If $m_l \; P_{m_l} \; w_k$, then m_l blocks π since $\pi_{m_l} \; P^{sd}_{m_l} \; \delta^{m_l m_l}_{m_l}$ is violated. If $w_k \; P_{m_l} \; m_l$, then by definition of R_{w_k}, $w_k \; P^{sd}_{w_k} \; m_l$ and w_k blocks π since $\pi_{w_k} \; P^{sd}_{w_k} \; \delta^{w_k w_k}_{w_k}$ is violated.

Let $\mu^1, \mu^2 \in \mathbb{M}$ be defined as follows:

μ^1	μ^2
$m_1 \leftrightarrow w_1$	$m_1 \leftrightarrow w_3$
$m_2 \leftrightarrow w_2$	$m_2 \leftrightarrow w_1$
$m_3 \leftrightarrow w_3$	$m_3 \leftrightarrow w_2$

If $\pi \in \Pi$ is such that for each $i \in N$, $\pi_i(i) = 1$, then $\pi \notin \mathcal{C}^{ssd}(R)$ since it is blocked by N through $\pi' \in \Pi$ defined by setting $\pi'(\mu^1) = \pi'(\mu^2) = \frac{1}{2}$. Thus, $\mathcal{C}^{ssd}(R) = \varnothing$.[5] ○

The last remaining set is the *expected utility stable matchings.* Before we prove existence of such matchings, we show that *stable* matchings are not necessarily included among them.

[5]Note that only coalitions of size 1 and $|N|$ are enough to block all $\pi \in \Pi$. Thus, the set of "strongly stochastic dominance individually rational and efficient" allocations is empty.

Example 6. *There are $R \in \mathcal{R}^N$ and $u \in \mathcal{U}(R)$ such that $S(R) \not\subseteq \mathcal{C}^{eu}(R)$.*

Consider the profile of preferences from Example 5. For each pair $i, j \in N$, let $u_i(j) = 10$ if j is i's top choice, $u_i(j) = 1$, if j is i's second choice, $u_i(j) = 0.99$ if j is i's third choice, and $u_i(j) = 0.98$ if j is i's fourth choice.

Note that $S(R) = \{\mu\}$ where for each $i \in N, \mu_i = i$. However, if $\pi \in \Pi$ is such that $\pi(\mu) = 1$, $\pi \notin \mathcal{C}^{eu}(u)$. To see this, we first note that for each $i \in N, U_i(\pi_i) = 1$. Consider $\pi' \in \Pi$ as defined in example 5. For each $i \in N, U_i(\pi_i) = 5.49$. Thus, N blocks π through π'. ◦

By appealing to a more general result in (Manjunath 2015) we have the following:

Proposition 6. *For each $R \in \mathcal{R}^N$ and $u \in \mathcal{U}(R)$, $\mathcal{C}^{eu}(u) \neq \varnothing$.*

While neither the *expected utility core* nor the *ex post core* are empty for any profile of preferences, the bad news is that there may not be an allocation that is in both sets. We see this in Example 2.

6 Conclusion

We have laid out several definitions of stability for the probabilistic extension of the marriage problem. However, one point of randomizing the marriage problem is to recover equity between the sexes. An intuitive way to do this is by randomizing within the deterministic stable set. The examples above show that doing this may provide *ex ante* incentives for groups, even just individuals or pairs, to reject the lottery in favor of a private (degenerate, for groups smaller than three) lotteries. The appropriate concept of the *core* or *stability* depends on the context of the model. The relations between the concepts presented here help with making that determination.

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