When too little is as good as nothing at all: Rationing a disposable good among satiable people with acceptance thresholds☆

Vikram Manjunath[1]

Abstract

We study the problem of rationing a divisible good among a group of people. Each person's preferences are characterized by an ideal amount that he would prefer to receive and a minimum quantity that he will accept: any amount less than this threshold is just as good as receiving nothing at all. Any amount beyond his ideal quantity has no effect on his welfare.

We search for *Pareto-efficient, strategy-proof,* and *envy-free* rules. The definitions of these axioms carry through from the more commonly studied problem without disposability or acceptance thresholds. However, these are not compatible in the model that we study. We adapt the *equal-division lower bound* axiom and propose another fairness axiom called *awardee-envy-freeness*. Unfortunately, these are also incompatible with *strategy-proofness*. We characterize all of the the *Pareto-efficient* rules that satisfy these two properties. We also characterize all *Pareto-efficient, strategy-proof,* and *non-bossy* rules. *JEL classification*: C71, D63

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1. Introduction

Imagine a town facing an energy shortage. Suppose that it has access to a fixed number of kilowatt hours of electricity. Town officials must divide these among local business owners. However, these owners have threshold quantities of electricity below which business is not viable. That is, if the quantity allocated to a particular business owner is lower than his threshold, it is as good as not allocating anything to him at all. Also, an owner is made better off as the quantity that he receives increases beyond his threshold, but only up to a certain level. We propose a model for such situations and study rules for making rationing decisions.

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[1]Département de Sciences Économiques, Université de Montréal

Another example is the allotment of time for using shared equipment. If sequencing does not matter and users care only about the length of time that they can use the equipment, our model is applicable: For each user's job, there is a minimum amount of time required to complete it. There is also a maximal amount of time he needs to complete his job, beyond which he has no use for the equipment.

Finally, if a good is being rationed among a group of people, each of whom incurs a fixed cost to use the good, then the quantity that a person requires to recoup his fixed cost is a natural lower bound on his acceptable quantities.

In our formal model, there is a social endowment to divide among a group of people. Each person in this group has preferences over the quantity that he receives. His preferences are described by a minimum threshold that he finds acceptable and an ideal amount that he would prefer to receive. He is indifferent between receiving any quantity below his threshold and receiving nothing at all. He finds a quantity above his threshold to be better and better as it increases, up to his ideal quantity. He is indifferent between any two quantities that are at least as high as his ideal amount. We do *not* require that the endowment be exhausted as it is disposable[2]. Indeed, there are situations where disposability is an appropriate assumption, even in the context of rationing: when some people have been satiated, the remainder may be unacceptable to those who have not.

If we restrict ourselves to rules that never give a person more than is needed to satiate him, an alternative interpretation of our model is that of a bankruptcy problem (O'Neill, 1982), where each claimant has some participation cost: if his award does not exceed his cost, he prefers not to show up and collect it. We interpret his ideal quantity as his claim.

We propose a set of axioms and search for rules that satisfy them. As usual, *Pareto-efficiency* says that no person can be made better off without this hurting another person. *Strategy-proofness*, also defined in the usual way, says that no person can benefit by unilaterally misrepresenting his preferences. Meaningful notions of fairness, on the other hand, are more difficult to define. One familiar notion requires that no person "envies" another. A weaker version is that people with identical preferences be treated identically. We show that no *Pareto-efficient* rule satisfies even the weaker of these two. We propose an axiom, *awardee-envy-freeness*, that says that no person who receives an amount that he finds acceptable envies any other person. This requirement *can* be met alongside *Pareto-efficiency*. We describe the class of all such rules. Another notion of fairness, called the *equal-division lower bound*, is that each person finds what he receives to be at least as desirable as an equal share of the social endowment. This requirement is compatible with *Pareto-efficiency* and we provide a necessary and sufficient condition for it to be met.

We also show that, unfortunately, every *Pareto-efficient* and *strategy-proof* rule violates both *awardee-envy-freeness* and the *equal-division lower bound*. We describe the class of all *Pareto-efficient, strategy-proof,* and *non-bossy* rules. Each of these rules always selects a very inequitable division.

Finally, we consider strengthening the strategic requirement to *group strategy-proofness*. This says that no group of people can misrepresent their preferences in a way that makes at least one member better off without hurting another member. We show that this axiom is incompatible even with very weak notions of efficiency.

[2] See Kıbrıs (2003) for another model where disposability is assumed.

For the closely related "classical" *division problem with single-peaked preferences*, the endowment is to be allocated in entirety. In contrast with our model, a single rule satisfies all of the axioms that we have discussed (Sprumont, 1991; Ching, 1992, 1994). In fact, this rule uniquely satisfies several other sets of axioms, some of which are discussed in the appendix (Thomson, 1994; Sönmez, 1994; Thomson, 1995; Schummer and Thomson, 1997; Thomson, 1997; Weymark, 1999). To re-iterate, the differences between this classical model and ours are free-disposal of the social endowment and acceptance thresholds.

The incompatibility of *Pareto-efficiency, strategy-proofness,* and *fairness* in our model is due to lower bounds on acceptable quantities. In a model without free-disposal, upper bounds on a person's consumption space are also meaningful. When both upper and lower bounds are introduced to the classical problem with *single-peaked preferences*, *strategy-proofness* is incompatible with even a weak notion of *efficiency* (Bergantiños et al., 2009). This is different from our model, where *strategy-proofness* and *efficiency* are compatible. When there are only upper bounds but no lower bounds, we no longer have any of the incompatibilities highlighted here. In fact, we obtain characterizations similar to the classical problem (Bergantiños et al., 2010).

A similarity between our model and one with both upper and lower bounds, without free-disposal, is that for a particular definition of *fairness*, the class of *Pareto-efficient* and *fair* rules (Bergantiños et al., 2009) resembles the class of *Pareto-efficient* and *awardee-fair* rules that we characterize for our model.

Though the social endowment is divisible, our results are comparable to those in the literature on problems where the social endowment is a set of indivisible objects (Pápai, 2001; Hatfield, 2009). This is because the lower bounds on acceptable quantities cause a sort of indivisibility of the good: either a person receives an acceptable quantity or nothing at all, but nothing in between.

2. The Model

Let $\boldsymbol{N}$ be a group of people and $\boldsymbol{M} \in \mathbb{R}_+$ be an amount of the divisible good. Each $i \in N$ has preferences over the quantity that is allotted to him. His preferences, $\boldsymbol{R_i}$ defined over $[0, \infty)$, are characterized by an acceptance threshold $\boldsymbol{l_i}$ and an ideal amount $\boldsymbol{p_i}$ such that $l_i \leq p_i$. He is indifferent between any two quantities $x, y \in [0, \infty)$ if either $x, y \leq l_i$ or $x, y \geq p_i$. In all other cases he prefers x to y whenever $x > y$. If i finds x to be at least as desirable as y under preference relation R_i, we write $x \; R_i \; y$. If he finds x to be more desirable than y, we write $x \; P_i \; y$ and $x \; I_i \; y$ if he is indifferent between them. Note that R_i is completely described by l_i and p_i.

Let $\boldsymbol{\mathcal{R}}$ be the set of all preference relations. A **problem** is a profile of preferences, $R \in \mathcal{R}^N$, and an amount of social endowment, $M \in \mathbb{R}_+$. The amount is to be allocated (not necessarily entirely) among N in such a way that the sum of what is awarded to each person does not exceed M. A **feasible allocation** at M is any vector $x \in \mathbb{R}_+^N$ such that $\sum_{i \in N} x_i \leq M$. Let $\boldsymbol{F(M)}$ denote the set of feasible allocations at M. Given $x \in F(M)$, for each $i \in N$, we denote i's component of x by x_i and the list of others' components by x_{-i}. Similarly, for each $R \in \mathcal{R}^N$ and each $i \in N$, let R_i denote i's preference relation and let R_{-i} denote the list of others' preferences. For each $R'_i \in \mathcal{R}$, let (R'_i, R_{-i}) be the profile where i has preference R'_i and the list of others' preferences is R_{-i}.

A **rule**, $\varphi : \mathcal{R}^N \times \mathbb{R}_+ \to \mathbb{R}_+^N$, associates every problem with a feasible allocation.

Remark 1. *An interesting subdomain:* Each problem where, for each $i \in N$ $l_i = 0$, can also be interpreted as classical problem with *single-peaked preferences* where the amount to divide does not exceed the sum of the peaks. The results for classical problems hold for this subdomain. ∘

3. Axioms

Let φ be a rule. The first requirement is that an allocation is chosen only if there is no other allocation that makes at least one person better off without making another worse off.

Pareto-efficiency: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, there is no $x \in F(M)$ such that, for each $i \in N, x_i \; R_i \; \varphi_i(R, M)$ and there is $i \in N$ such that $x_i \; P_i \; \varphi_i(R, M)$.

A weakening of *Pareto efficiency* is that an allocation is chosen only if there is no other allocation that makes *every* person better off.

Weak Pareto-efficiency: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, there is no $x \in F(M)$ such that, for each $i \in N, x_i \; P_i \; \varphi_i(R, M)$.

A further weakening says that if the social endowment equals the sum of the ideal amounts, then each person should receive his ideal amount.[3]

Unanimity: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, if $\sum_{i \in N} p_i = M$, then $\varphi(R, M) = (p_i)_{i \in N}$.

The next property is that no person can benefit by misreporting his preferences. In the following definition, and later in the proofs of our results, given $i \in N$ and a pair $R_i, R'_i \in \mathcal{R}$, we place a T above i's "true" preference relation $(\overset{\text{T}}{R_i})$, and an F above a "false" preference relation $(\overset{\text{F}}{R'_i})$.

Strategy-proofness: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, and each $i \in N$, there is no $R'_i \in \mathcal{R}$, such that $\varphi_i(\overset{\text{F}}{R'_i}, R_{-i}, M) \; \overset{\text{T}}{P_i} \; \varphi_i(\overset{\text{T}}{R_i}, R_{-i}, M)$.

A more demanding property is that no group of people can misreport in a way that makes at least one of its members better off without making any of its members worse off.

Group strategy-proofness: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ and each $S \subseteq N$, there is no $R'_S \in \mathcal{R}^S$ such that for each $i \in S, \varphi_i(\overset{\text{F}}{R'_S}, R_{-S}, M) \; \overset{\text{T}}{R_i} \; \varphi_i(\overset{\text{T}}{R_S}, R_{-S}, M)$ and there is $i \in S$ such that $\varphi_i(\overset{\text{F}}{R'_S}, R_{-S}, M) \; \overset{\text{T}}{P_i} \; \varphi_i(\overset{\text{T}}{R_S}, R_{-S}, M)$.

It turns out that *group strategy-proofness* is "too demanding" in the sense that it is not compatible with even *unanimity*, the weakest of our efficiency notions.

Next we present two notions of fairness. First, no person "envies" another.

Envy-freeness: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ there is no pair $i, j \in N$ such that $\varphi_j(R, M) \; P_i \; \varphi_i(R, M)$.

The next is that people with identical preferences are treated identically.

Equal treatment of equals: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, and each pair $i, j \in N$ such that $R_i = R_j = R_0$, we have $\varphi_i(R, M) \; I_0 \; \varphi_j(R, M)$.

Envy-freeness implies *equal-treatment of equals*. We will show that, even *equal treatment of equals* is incompatible with *Pareto-efficiency*. We propose another weakening of

[3] A stronger, but more appealing, version says that each person receives his ideal amount if the social endowment is *at least as large as* the sum of the ideal amounts.

envy-freeness that applies only to those people who receive an acceptable share. This axiom can be interpreted as follows: after the social endowment is divided each person shows up to receive his share only if he finds it acceptable and no person who shows up envies any other person.

Awardee-envy-freeness: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, there is no pair $i, j \in N$ such that $\varphi_i(R, M) > l_i$ and $\varphi_j(R, M)\ P_i\ \varphi_i(R, M)$.

Our final notion of fairness is that each person receives an amount that he finds at least as desirable as an equal share of the social endowment.

Equal-division lower bound:[4] For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, and each $i \in N$, $\varphi_i(R, M)\ R_i\ \frac{M}{|N|}$.[5]

If, at an allocation $x \in F(M)$, there is $i \in N$ who receives an unacceptable amount, then he is indifferent between receiving x_i and receiving 0. So, he either does not take it, or takes it and disposes of it. The planner may as well dispose of x_i units of the good himself and not give i anything. Similarly, when $x_i > p_i$, the planner may as well dispose of $x_i - p_i$ units of the good and give i only p_i. This is expressed by the next axiom.

For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, define the **welfare-equivalence class of $\boldsymbol{x}$ at $\boldsymbol{(R, M)}$, $\boldsymbol{WE(x, R, M)}$**, by setting $WE(x, R, M) \equiv \{y \in F(M) : \text{ for each } i \in N, x_i\ I_i\ y_i\}$. If $WE(x, R, M) = \{x\}$, then **$\boldsymbol{x}$ is welfare-unique at $\boldsymbol{(R, M)}$**. For each $x \in F(M)$ such that x is not *welfare-unique at* (R, M), there is $x' \in WE(x, R, M)$ such that for each $i \in N$, if $x_i \leq l_i$ then $x'_i = 0$, if $x_i > p_i$ then $x'_i = p_i$, and otherwise $x'_i = x_i$. Such x' is the **canonical representation of $\boldsymbol{WE(x, R, M)}$ at $\boldsymbol{(R, M)}$**. If x is either *welfare-unique* or the *canonical representation of* $WE(x, R, M)$ *at* (R, M), then x is **canonical at $\boldsymbol{(R, M)}$**.

Canonicity: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, $\varphi(R, M)$ is *canonical at* (R, M).

The next requirement is an application of the "replacement principle" (Moulin, 1987; Thomson, 1997)[6]. If the preferences of one person change, then all others are affected in the same direction: either each person finds his new share at least as desirable as his original share, or each person finds his original share to be at least as desirable as his new share.

Welfare-dominance under preference replacement: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}$, each $i \in N$ and each $R'_i \in \mathcal{R}$, either for each $j \in N \setminus \{i\}, \varphi_j(R'_i, R_{-i}, M)\ R_j\ \varphi_j(R_i, R_{-i}, M)$ or for each $j \in N \setminus \{i\}, \varphi_j(R_i, R_{-i}, M)\ R_j\ \varphi_j(R'_i, R_{-i}, M)$.

The following requirement is that if a person's preferences change in a way that his own component of the allocation remains the same, then others' components should also remain the same (Satterthwaite and Sonnenschein, 1981). While it appears technical, this axiom is normatively appealing since it is implied by the combination of the previous two axioms along with *Pareto-efficiency*.

Another interpretation of this axiom is that, alongside *canonicity*, it precludes a particular kind of profitable misreporting by pairs of people: one person misreports his

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[4] This axiom is often referred to as "individual rationality from equal division."

[5] Analogous to *awardee envy-freeness*, an "awardee" version of this axiom can be defined as follows: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, for each $i \in N$, if $\varphi_i(R, M) > l_i$, then $\varphi_i(R, M)\ R_i\ \frac{M}{|\{j \in N : \varphi_j(R, M) > l_j\}|}$. Not only is the class of *Pareto-efficient* and *awardee envy-free* rules meeting the *equal-division lower bound* non-empty, but every member satisfies the above axiom. Therefore, we do not study it (even though it is not directly implied by the *equal-division lower bound*).

[6] For a survey, see Thomson (1999).

preferences in a way that leaves his share unaffected, but another person is made better off. This is still is weaker than *group strategy-proofness* since it does not even imply *strategy-proofness*.

Non-bossiness: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, each $i \in N$ and each $R'_i \in \mathcal{R}$, if $\varphi_i(R'_i, R_{-i}, M) = \varphi_i(R_i, R_{-i}, M)$ then $\varphi_{-i}(R'_i, R_{-i}, M) = \varphi_{-i}(R_i, R_{-i}, M)$.

Our final axiom is that small changes in the data of the problem should cause only small changes in the chosen allocation. Let $\{(R^\nu, M^\nu)\}_{\nu=1}^{\infty}$ be a sequence in $\mathcal{R}^N \times \mathbb{R}_+$ with limit (R, M).[7]

Continuity: $\lim_{\nu\to\infty} \varphi(R^\nu, M^\nu) = \varphi(R, M)$.

We will show that no *Pareto-efficient* rule is *continuous*. The set of *Pareto-efficient* allocation is ill behaved with regards to certain sequences: those where at least one person's preference relation is complete indifference only at the limit. We propose a weaker version of *continuity* which ignores the aforementioned problematic sequences and is compatible with *Pareto-efficiency*.

Weak continuity: If for each $i \in N$, for whom $l_i \geq \min\{p_i, M\}$, there is $\nu^* \in \mathbb{N}$ such that for each $\nu \geq \nu^*, l_i^\nu \geq \min\{p_i^\nu, M^\nu\}$, then $\lim_{\nu\to\infty} \varphi(R^\nu, M^\nu) = \varphi(R, M)$.

4. Rules

Each member of the first class of rules that we describe is associated with an ordering over people. The person with highest priority takes the least of his peak and the endowment as long as it is acceptable to him. Then, the person with second highest priority takes the least of his peak and what remains as long as it is acceptable to him. We then proceed to the person with third highest priority, and so on.[8]

Let I be the ordering $i_1, \ldots, i_n$ of N. The **sequential priority rule with respect to $\boldsymbol{I}, \boldsymbol{SD^I}$,** is defined by setting for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, $SP^I(R, M) \equiv x$ such that for each $k = 1 \ldots, n$,

$$x_{i_k} \equiv \begin{cases} \min\{M - \sum_{l=1}^{k-1} x_{i_l}, p_{i_k}\} & \text{if } \min\{M - \sum_{l=1}^{k-1} x_{i_l}, p_{i_k}\} > l_{i_k}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

These Rules are *Pareto-efficient*, *strategy-proof*, *welfare-dominant under preference replacement*, *non-bossy*, *canonical*, and *weakly continuous*. However, they violate *awardee-envy-freeness*, the *equal-division lower bound*, *group strategy-proofness*, and *continuity*.

This class can be expanded by considering "conditional" *priority orders*: The identity of the person with highest priority depends on the social endowment. The identity of the person with second highest priority depends on the endowment, the identity of the first person, and the quantity that he has taken. The identity of the third person depends on the endowment, the identities of the first two people, and the quantities that they have taken, and so on.

Members of this bigger class satisfy all of the same axioms as *sequential priority*, save for *weak continuity*.

[7]Since each preference relation $R_i \in \mathcal{R}$ is uniquely identified by a pair $(l_i, p_i) \in \mathbb{R}_+$, this is a sequence in $\mathbb{R}^{2n+1}$.

[8]Members of this family of rules are similar to those proposed by Pápai (2001) and also studied by Ehlers and Klaus (2003) and Hatfield (2009) for the problem of assigning multiple objects.

Let $I \equiv \{i^k\}_{k=1}^n$, where for each $k = 1, \ldots, n, i^k : \mathbb{R}_+ \times N^{k-1} \times \mathbb{R}_+^{k-1} \to N$, be such that for each $M \in \mathbb{R}_+$ and each $x \in \mathbb{R}^{n-1}$, we have a sequence $\{i_k\}_{k=1}^n$ such that for each $k = 1, \ldots, n$, $i_k = i^k(M, i_1, \ldots, i_{k-1}, x_{i_1}, \ldots, x_{i_{k-1}}) \notin \{i_1, \ldots, i_{k-1}\}$. The **conditional sequential priority rule with respect to $\boldsymbol{I}, \boldsymbol{CSP^I}$,** is defined by setting for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, $CSP^I(R, M) \equiv x$ such that, for each $k = 1, \ldots, n, i_k \equiv i_k = i^k(M, i_1, \ldots, i_{k-1}, x_{i_1}, \ldots, x_{i_{k-1}})$ and

$$x_{i_k} \equiv \begin{cases} \min\{M - \sum_{l=1}^{k-1} x_{i_l}, p_{i_k}\} & \text{if } \min\{M - \sum_{l=1}^{k-1} x_{i_l}, p_{i_k}\} > l_{i_k}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The next rule is analogous to the well-known "uniform rule" for the classical problem with *single-peaked preferences* (Bénassy, 1982; Sprumont, 1991). Define the **uniform rule, $\boldsymbol{U}$**, by setting, for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, and $i \in N$,

$$U_i(R, M) \equiv \begin{cases} p_i & \text{if } \sum_{i \in N} p_i \leq M, \text{ or} \\ \min\{p_i, \lambda\} & \text{otherwise,} \end{cases}$$

where λ is such that $\sum_{i \in N} U_i(R, M) = M$.[9]

While the *uniform* rule is *weakly Pareto-efficient, strategy-proof, envy-free, welfare-dominant under preference replacement, non-bossy,* and *continuous*, it is not *Pareto-efficient*. In contrast with the corresponding rule for the classical setting, this rule is not *group strategy-proof*.[10]

Members of the next class of rules are *Pareto-efficient*. Unfortunately, they are neither *strategy-proof* nor *envy-free*.

Before we introduce this class of rules, define for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, the **efficient uniform coalitions at $\boldsymbol{(R, M)}, \boldsymbol{EUC(R, M)}$,** by setting,

$$EUC(R, M) \equiv \left\{ N' \subseteq N : \begin{array}{lll} \text{there is} & i) & \text{for each } j \notin N', M - \sum_{i \in N'} \min\{p_i, \lambda\} \leq l_j, \\ \lambda \in \mathbb{R}_+ & & \text{and} \\ \text{such that} & ii) & \text{for each } j \in N', \min\{p_j, \lambda\} > l_j. \end{array} \right\}.$$

We show that $EUC(R, M) \neq \varnothing$.

Claim 1. *For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, $EUC(R, M) \neq \varnothing$.*

Proof: Suppose there is $i \in N$ for whom $l_i < \min\{p_i, M\}$. Let $\{1, 2, \ldots, k\} \subseteq N$ be such that $l_1 \leq l_2 \leq \cdots \leq l_k < M$. By assumption, $k \geq 1$. Start with $\{1\}$. If $M - \min\{p_1, M\} \leq l_2$, then $\{1\} \in EUC(R, M)$. Otherwise, $\{1\} \notin EUC(R, M)$. Suppose that $\{1, 2, \ldots, j\} \notin EUC(R, M)$. Then $M - \sum_{i=1}^j \min\{p_i, M\} > l_{j+1}$. If $\{1, \ldots, k\} \notin EUC(R, M)$, then $k \neq n$ and $M - \sum_{i=1}^k p_i > l_{k+1} \geq M$, which is a contradiction. On the other hand, if for each $i \in N$, $l_i \geq \min\{p_i, M\}$, then $EUC(R, M) = \{\varnothing\}$. △

[9] Of course, the *uniform* rule is the logical (though, not always obvious) extension of the *constrained equal awards* rule from bankruptcy problems to the division problem with *single-peaked preferences*. The model studied here is, in some senses, between the two and we have chosen to fall on the *uniform rule* side of the nomenclature fence.

[10] To see this, let $R \in \mathcal{R}^N$ and $i \in N$ be such that $0 < \lambda \leq l_i$ and $j \in N \setminus \{i\}$ such that $\lambda \in (l_j, p_j)$. Let $R'_i \in \mathcal{R}$ be such that $p'_i = 0$. Then, $0 = \varphi_i(\overset{\text{F}}{R'_i}, R_{-i}) \overset{\text{T}}{I_i} \varphi_i(\overset{\text{T}}{R_i}, R_{-i}) = \lambda$ and since $\varphi_j(\overset{\text{F}}{R'_i}, R_{-i}) > \lambda > l_j$, $\varphi_j(\overset{\text{F}}{R'_i}, R_{-i}) \overset{\text{T}}{P_j} \varphi_j(\overset{\text{T}}{R_i}, R_{-i})$.

A **selector** is a function, $\sigma : \mathcal{R}^N \times \mathbb{R}_+ \to \mathbb{P}(N)$,[11] such that for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, $\sigma(R, M) \in EUC(R, M)$. The **efficient uniform rule with selector $\boldsymbol{\sigma}$, $\boldsymbol{EU^\sigma}$,** is defined by setting, for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ and each $i \in N$,

$$EU_i^\sigma(R, M) \equiv \begin{cases} \min\{p_i, \lambda\} & \text{if } i \in \sigma(R, M), \text{ or} \\ 0 & \text{otherwise.} \end{cases}$$

where λ is such that:

i) for each $j \notin \sigma(R, M), M - \sum_{i \in \sigma(R,M)} \min\{p_i, \lambda\} \leq l_j$,
ii) for each $j \in \sigma(R, M), \lambda \geq l_j$, and
iii) if $\sum_{i \in \sigma(R,M)} \min\{p_i, \lambda\} < M$ then $\lambda \geq \max_{i \in \sigma(R,M)}\{p_i\}$.

Efficient uniform rules are *Pareto-efficient, awardee-envy-free,* and *non-bossy*. They are not, however, *strategy-proof, welfare-dominant under preference replacement,* or *envy-free*. In fact, as we will show, no *Pareto-efficient* rule is *envy-free*. Only some members of this class are *weakly continuous*.

5. Results

We begin with a necessary and sufficient condition for a rule to meet the *equal-division lower bound*.

Proposition 1. *A rule, φ, meets the* equal-division lower bound *if and only if for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, for each $i \in N$ if $l_i < \frac{M}{|N|}$ then $\varphi_i(R, M) \geq \min\left\{p_i, \frac{M}{|N|}\right\}$.*

Proof: Let $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ and $i \in N$ be such that $l_i < \frac{M}{|N|}$. By the *equal-division lower bound*, $\varphi_i(R, M)\ R_i\ \frac{M}{|N|}$. If $\varphi_i(R, M) < \frac{M}{|N|}$, then $\varphi_i(R, M)\ I_i\ \frac{M}{|N|}\ I_i\ p_i$. Thus, $\varphi_i(R, M) \geq p_i$. If $\varphi_i(R, M) < p_i$, then, since $\varphi_i(R, M)\ R_i\ \frac{M}{|N|}$, $\varphi_i(R, M) \geq \frac{M}{|N|}$.

Conversely, if for each (R, M) and each $i \in N$ such that $l_i < \frac{M}{|N|}$, $\varphi_i(R, M) \geq \min\{p_i, \frac{M}{|N|}\}$, then $\varphi_i(R, M)\ R_i\ \frac{M}{N}$. For each $i \in N$ such that $l_i \geq \frac{M}{|N|}$, $\frac{M}{|N|}\ I_i\ 0$, so $\varphi_i(R, M)\ R_i\ 0\ I_i\ \frac{M}{|N|}$. Thus, φ satisfies the *equal-division lower bound*. □

Each rule is equivalent in "welfare terms" to a *canonical* rule. For each rule φ, define its **canonical equivalent,** φ', by setting, for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, and for each $i \in N, \varphi_i'(R, M) \equiv 0$ if $\varphi_i(R, M) \leq l_i, \varphi_i'(R, M) \equiv p_i$ if $\varphi_i(R, M) > p_i$, and $\varphi_i'(R, M) \equiv \varphi_i(R, M)$ otherwise.

Since *Pareto-efficiency, envy-freeness, awardee-envy-freeness,* and *the equal-division lower bound* deal with a single preference profile and are stated in welfare terms, these properties are inherited by a rule's *canonical equivalent*. It is not as obvious that *strategy-proofness* is inherited.

Lemma 1. *The* canonical equivalent *of a* strategy-proof *rule is* strategy-proof.[12]

[11] Given a set A, let $\mathbb{P}(A)$ be the power set of A.

[12] The converse is not true. Consider φ such that, for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, $\varphi_1(R, M) = M$ if $p_1 \leq \frac{M}{2}$ and $\varphi_1(R, M) = \frac{M}{2}$ otherwise, and $\varphi_{-1}(R, M) = (0, \ldots, 0)$. Then φ's *canonical equivalent* is φ' such that for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, $\varphi_1'(R, M) = \max\{p_1, \frac{M}{2}\}$ if $l_1 < \frac{M}{2}$ and $\varphi_1'(R, M) = 0$ otherwise, and $\varphi_{-1}(R, M) = (0, \ldots, 0)$. Though φ' is *strategy-proof, φ is not.*

Proof: Let φ' be the canonical equivalent of φ. Then, for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ and each $i \in N$,

$$\begin{array}{rcll} \varphi_i(R,M) & = & \varphi_i'(R,M) & \text{if } l_i < \varphi_i(R,M) < p_i. \\ \varphi_i(R,M) & \geq & \varphi_i'(R,M) & \text{otherwise.} \end{array}$$

Suppose that φ' is not *strategy-proof*. Then, there is $i \in N$, and $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ such that

$$\varphi_i'(\overset{\text{F}}{R_i'}, R_{-i}, M) \overset{\text{T}}{P_i} \varphi_i'(\overset{\text{T}}{R_i}, R_{-i}, M).$$

Then, $\varphi_i'(R, M) < p_i$. If $\varphi_i'(R, M) > l_i$, then $\varphi_i(R, M) = \varphi_i'(R, M)$ and if $\varphi_i'(R, M) = 0$, then $\varphi_i(R, M) \leq l_i$. Thus,

$$\max\{\varphi_i'(R, M), l_i\} \geq \varphi_i(R, M). \tag{1}$$

Further,

$$\varphi_i'(R_i', R_{-i}, M) > \max\{\varphi_i'(R, M), l_i\}. \tag{2}$$

Since $\varphi_i(R_i', R_{-i}, M) \geq \varphi_i'(R_i', R_{-i}, M)$, by (1) and (2),

$$\varphi_i(R_i', R_{-i}, M) \geq \varphi_i'(R_i', R_{-i}, M) > \max\{\varphi_i'(R, M), l_i\} \geq \varphi_i(R, M).$$

Then, $\varphi_i(\overset{\text{F}}{R_i'}, R_{-i}, M) \overset{\text{T}}{P_i} \varphi_i(\overset{\text{T}}{R_i}, R_{-i}, M)$. So φ is not *strategy-proof.* □

In light of Lemma 1, it is without loss of generality to study only *canonical* rules. The characterizations that follow are thus in welfare terms.

Theorem 1. *A rule is* Pareto-efficient *and* awardee-envy-free *if and only if it is an* efficient uniform rule.

Proof: Let φ be a *Pareto-efficient* and *awardee-envy-free* rule. Let σ^φ be the *selector* defined by setting, for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$,

$$\sigma^\varphi(R, M) \equiv \{i \in N : \varphi_i(R, M) \neq 0\}.$$

Let $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ and $x \equiv \varphi(R, M)$. Suppose that $x \neq EU^{\sigma^\varphi}(R, M)$.
Case 1: There is a pair $i, j \in N$ such that $0 < x_i < x_j$ and $x_i < p_i$. In this case, $x_j \; P_i \; x_i$. This violates *awardee-envy-freeness*.
Case 2: There is j such that $x_j = 0$ and $M - \sum_{i \in N} x_i > l_j$. Let $y \in \mathbb{R}^N_+$ be such that $y_j = M - \sum_{i \in N} x_i$ and $y_{-j} = x_{-j}$. For each $i \in N, y_i \; R_i \; x_i$ and $y_j \; P_j \; x_j$. This violates *Pareto-efficiency*.
Case 3: There is $j \in N$ such that $x_j \in (0, l_j]$. This violates *canonicity*.
Case 4: There is $j \in N$ such that $l_j < x_j < p_j$ and $\sum_{i \in N} x_i < M$. Let $y \in \mathbb{R}^N_+$ be such that $y_j = x_j + M - \sum_{i \in N} x_i$ and $y_{-j} = x_{-j}$. For each $i \in N, y_i \; R_i \; x_i$ and $y_j \; P_j \; x_j$. This violates *Pareto-efficiency*.

Thus, $\varphi(R, M) = EU^{\sigma^\varphi}(R, M)$. □

The axioms are independent. The *canonical equivalent* of the *uniform* rule is *awardee-envy-free* but not *Pareto-efficient*, and that the *sequential priority* rules are *Pareto-efficient* but not *awardee-envy-free*.

Combining Proposition 1 and Theorem 1, we have the following corollary.

Corollary 1. *A rule* φ *is* Pareto-efficient, awardee-envy-free, *and satisfies the* equal-division lower bound *if and only if there is a selector* σ *such that for each* $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$,

$$\left\{ i \in N : l_i < \frac{M}{|N|} \right\} \subseteq \sigma(R, M),$$

and $\varphi = EU^\sigma$.

To prove the next characterization, we first establish that for each $M \in \mathbb{R}_+$, a *Pareto-efficient, strategy-proof,* and *non-bossy* rule assigns the highest priority to a person who, regardless of others' preferences, receives the least of his peak and the endowment as long as it is acceptable to him.

Lemma 2. *Let* φ *be a* Pareto-efficient, strategy-proof, *and* non-bossy *rule. Then, for each* $M \in \mathbb{R}$, *there is* $i \in N$ *such that for each* $R \in \mathcal{R}^N$,

$$\varphi_i(R, M) = \begin{cases} \min\{M, p_i\} & \textit{if } \min\{M, p_i\} > l_i \\ 0 & \textit{otherwise.} \end{cases}$$

Proof: We prove this lemma by induction on $n \equiv |N|$.

We start with $n = 2$. Let $R \in \mathcal{R}^N$ be such that $l_1 = l_2 = 0$ and $p_1 = p_2 = M$. Let $x \equiv \varphi(R, M)$. By *Pareto-efficiency*, $x_1 + x_2 = M$.
Claim: Either $x_1 = M$ or $x_2 = M$.
Proof: If not, $x_1, x_2 \in (0, M)$. Let $R' \in \mathcal{R}^N$ be such that $l'_1 = x_1$, $l'_2 = x_2$, and $p'_1 = p'_2 = M$. Let $x^1 \equiv \varphi(R'_1, R_2, M)$. By *strategy-proofness*, $x^1_1 \leq x_1$: Otherwise $x^1_1 = \varphi_1(\overset{\text{F}}{R'_1}, R_2, M) \; \overset{\text{T}}{P_1} \; \varphi(\overset{\text{T}}{R_1}, R_2, M)$. By *canonicity* and since $x^1_1 \leq x_1 = l'_1$, we deduce that $x^1_1 = 0$. By *Pareto-efficiency*, $x^1_2 = M$. That is, $\varphi(R'_1, R_2, M) = (0, M)$. By an analogous argument, $\varphi(R_1, R'_2, M) = (M, 0)$.

Let $x' \equiv \varphi(R', M)$. Since $x_1 + x_2 = M$, we cannot have $x'_1 \geq l'_1 = x_1$ and $x'_2 \geq l'_2 = x_2$. By *Pareto-efficiency*, either $x' = (0, M)$ or $x' = (M, 0)$. If $x' = (0, M)$, since $M = \varphi_1(\overset{\text{F}}{R_1}, R'_2, M) \; \overset{\text{T}}{P'_1} \; \varphi_1(\overset{\text{T}}{R'_1}, R'_2, M) = 0$, this violates *strategy-proofness*. If $x' = (M, 0)$, we reach a similar contradiction. This establishes our claim.

Let $i \in N$ be such that $x_i = M$. For each $R'_i \in \mathcal{R}$ such that $l'_i < \min\{M, p'_i\}$, by *strategy-proofness*, $\varphi(R'_i, R_j, M) \geq \min\{M, p'_i\}$: otherwise, $M = \varphi_i(\overset{\text{F}}{R_i}, R_j, M) \; \overset{\text{T}}{P'_i} \; \varphi_i(\overset{\text{T}}{R'_i}, R_j, M)$. By *Pareto-efficiency*, $\varphi_i(R'_i, R_j, M) \leq \min\{M, p'_i\}$. Thus, we have $\varphi_i(R'_1, R_j, M) = \min\{M, p'_i\}$ and $\varphi_j(R'_i, R_j, M) = M - \min\{M, p'_i\}$. By *strategy-proofness*, for each $R'_j \in \mathcal{R}$, $\varphi_j(R'_i, R'_j, M) \leq M - \min\{M, p'_i\}$: otherwise we have $\varphi_j(R'_i, \overset{\text{F}}{R'_j}, M) \; \overset{\text{T}}{P_j} \; \varphi_j(R_i, \overset{\text{T}}{P_j}, M) = M - \min\{M, p'_i\}$. By *Pareto-efficiency* and *canonicity*, $\varphi_i(R'_i, R'_j, M) = \min\{M, p'_i\}$. If $\min\{M, p'_i\} \leq l'_i$, by *canonicity*, $\varphi_i(R', M) = 0$.

Thus, for each $M \in \mathbb{R}$, there is $i \in N$ such that for each $R \in \mathcal{R}^N$,

$$\varphi_i(R, M) = \begin{cases} \min\{M, p_i\} & \text{if } \min\{M, p_i\} > l_i \\ 0 & \text{otherwise.} \end{cases}$$

As an *induction hypothesis*, suppose that if $|N'| = n - 1$ and ψ is a *Pareto-efficient, strategy-proof,* and *non-bossy* rule defined for N', then for each M, there is $i \in N'$ such that for each $R \in \mathcal{R}^{N'}$,

$$\psi_i(R, M) \;=\; \begin{cases} \min\{M, p_i\} & \text{if } \min\{M, p_i\} > l_i \\ 0 & \text{otherwise.} \end{cases}$$

We show that for each $M \in \mathbb{R}$, there is $i \in N$ such that for each $R \in \mathcal{R}^N$,

$$\varphi_i(R, M) \;=\; \begin{cases} \min\{M, p_i\} & \text{if } \min\{M, p_i\} > l_i \\ 0 & \text{otherwise.} \end{cases}$$

Let $R \in \mathcal{R}^N$ be such that for each $i \in N$, $l_i = 0$ and $p_i = M$. Let $x \equiv \varphi(R, M)$. Since φ is *Pareto-efficient*, $\sum_{j \in N} x_j = M$.
Claim: There is $i \in N$ such that $x_i = M$.
Proof:
Case 1: There is $j \in N$ such that $x_j = 0$.

Let $\tilde{R}_j \in \mathcal{R}$ be such that $\tilde{p}_j = 0$. By *canonicity*, for each $\overline{R}_{-j} \in \mathcal{R}^{N \setminus \{j\}}$, and each $\overline{M} \in \mathbb{R}_+$, $\varphi_j(\tilde{R}_j, \overline{R}_{-j}, \overline{M}) = 0$. By *non-bossiness*,[13] $\varphi(\tilde{R}_j, R_{-j}, M) = x$. Now, define ψ as a rule for $N \setminus \{j\}$ by setting, for each $(\overline{R}_{-j}, \overline{M}) \in \mathcal{R}^{N \setminus \{j\}} \times \mathbb{R}$,

$$\psi(\overline{R}_{-j}, \overline{M}) \equiv \varphi_{-j}(\tilde{R}_j, \overline{R}_{-j}, \overline{M}).$$

Since φ is *Pareto-efficient, strategy-proof*, and *non-bossy*, ψ inherits these properties. By the *induction hypothesis*, there is $i \in N \setminus \{j\}$ such that $\psi_i(R_{-j}, M) = M$. However, by definition of ψ, we have $\psi(R_{-j}, M) = x_{-j}$. Thus, there is $i \in N$ such that $x_i = M$.
Case 2: For each $j \in N, x_j \in (0, M)$.

Let $R' \in \mathcal{R}^N$ be such that for each $j \in N, x_j = l'_j < p'_j = M$. For each $j \in N$, define $x^j \equiv \varphi(R'_j, R_{-j}, M)$. By *strategy-proofness*, $x^j_j \leq x_j = l'_j$: Otherwise $x^j_j = \varphi_j(\overset{\text{F}}{R'_j}, R_{-j}, M) \; \overset{\text{T}}{R_j} \; \varphi_j(\overset{\text{T}}{R_j}, R_{-j}, M) = x_j$. By *canonicity*, $x^j_j = 0$. By an argument identical to the one made in Case 1, there is $k^j \in N \setminus \{j\}$ such that $x^j_{k^j} = M$.

Let $x' \equiv \varphi(R', M)$. Since $\sum_{j \in N} x_j = M$, there is $j \in N$ such that $x'_j \leq x_j = l'_j$. By *canonicity*, $x'_j = 0$. By an argument identical to the one made in Case 1, there is $i \in N$ such that $x'_i = M$. Thus, for each $j \in N \setminus \{i\}$, $\varphi_j(R', M) = 0$. In particular,

$$\varphi_{k^i}(R'_i, R'_{k^i}, R'_{-\{i,k^i\}}, M) = 0. \tag{3}$$

As shown earlier, $\varphi(R'_i, R_{k^i}, R_{-\{i,k^i\}}, M) = x^i$ and $x^i_{k^i} = M$. Thus, for each $j \in N \setminus \{i, k^i\}$, $\varphi_j(R'_i, R_{k^i}, R_{-\{i,k^i\}}, M) = 0$. Let $j_1, \ldots, j_{n-2}$ be a labeling of $N \setminus \{i, k^i\}$. By *strategy-proofness*, $\varphi_{j_1}(R'_i, R_{k^i}, R'_{j_1}, R_{-\{i,k^i,j_1\}}, M) = 0$: Otherwise, $\varphi_{j_1}(R'_i, R_{k^i}, \overset{\text{F}}{R'_{j_1}}, R_{-\{i,k^i,j_1\}}, M) \; \overset{\text{T}}{P_{j_1}} \; \varphi_{j_1}(R'_i, R_{k^i}, \overset{\text{T}}{R_{j_1}}, R_{-\{i,k^i,j_1\}}, M) = 0$. By *non-bossiness*, $\varphi(R'_i, R_{k^i}, \overset{\text{F}}{R'_{j_1}}, R_{-\{i,k^i,j_1\}}, M) = x^i$. Repeating this argument $n{-}3$ times, $\varphi(R'_i, R_{k^i}, R'_{-\{i,k^i\}}, M) = x^i$. Thus, $\varphi_{k^i}(R'_i, R_{k^i}, R'_{-\{i,k^i\}}, M) = M$. By (3) and the definition of R'_{k_i},

$$M = \varphi_{k^i}(R'_i, \overset{\text{F}}{R_{k^i}}, R'_{-\{i,k^i\}}, M) \; \overset{\text{T}}{P'_{k^i}} \; \varphi_{k^i}(R'_i, \overset{\text{T}}{R'_{k^i}}, R'_{-\{i,k^i\}}, M) = 0.$$

[13] This is the first time we appeal to *non-bossiness*.

This violates *strategy-proofness*. Thus, there is $i \in N$ such that $x_i = M$. This establishes our claim.

Let $R' \in \mathcal{R}^N$. To complete the proof of this lemma, we show that

$$\varphi_i(R', M) = \begin{cases} \min\{p'_i, M\} & \text{if } \min\{p'_i, M\} > l'_i \\ 0 & \text{otherwise.} \end{cases}$$

By *Pareto-efficiency*, if $\min\{p'_i, M\} \le l'_i$, then $\varphi_i(R', M) = 0$. If not, we index the members of $N \setminus \{i\}$ as $\{j_1, \ldots, j_{n-1}\}$. By *strategy-proofness*, $\varphi_{j_1}(R'_{j_1}, R_{-j_1}, M) = 0$: Otherwise, $\varphi_{j_1}(\overset{F}{R'_{j_1}}, R_{-j_1}, M)\ \overset{T}{P_{j_1}}\ \varphi(\overset{T}{R_{j_1}}, R_{-j_1}, M) = 0$. Then, by *non-bossiness*, $\varphi(R'_{j_1}, R_{-j_1}, M) = x$. Repeating this argument $n-2$ times, $\varphi(R'_{-i}, R_i, M) = x$. That is, $\varphi_i(R'_{-i}, R_i, M) = M$. By *strategy-proofness*, $\varphi_i(R', M) \ge \min\{M, p'_i\}$: Otherwise, $M = \varphi(\overset{F}{R_i}, R'_{-i}, M)\ \overset{T}{P'_i}\ \varphi_i(\overset{T}{R'_i}, R'_{-i}, M)$. Due to the feasibility constraint, $\varphi_i(R', M) \le M$ and by *canonicity*, $\varphi_i(R', M) \le p'_i$. Thus, $\varphi_i(R', M) = \min\{M, p'_i\}$. □

Lemma 2 gets us very close to a characterization of the *conditional sequential priority* rules as the only *Pareto-efficient, strategy-proof,* and *non-bossy* rules. It says that for each endowment, there is a person with highest priority who gets the least of his most preferred quantities. After this, if we are left with more of the good to divide among the remaining people, Lemma 2 can be applied again.

Theorem 2. *A rule is* Pareto-efficient, strategy-proof, *and* non-bossy *if and only if it is a* conditional sequential priority rule.

Proof: We proceed by induction on $n = |N|$. If $n = 2$, the theorem follows directly from Lemma 2. As an *induction hypothesis*, suppose the result holds for N' such that $|N'| = n - 1$.

Let φ be a *Pareto-efficient, strategy-proof,* and *non-bossy* rule defined for people in N such that $|N| = n$. We will show that there is I_φ such that $\varphi = CSP^{I_\varphi}$.

Step 1: Construction of $I_\varphi \equiv \{i^k_\varphi\}_{k=1}^{n-2}$.

By Lemma 2, for each $M \in \mathbb{R}$, there is $i_1 \in N$ such that for each $R \in \mathcal{R}^N$,

$$\varphi_{i_1}(R, M) = \begin{cases} \min\{p_{i_1}, M\} & \text{if } \min\{p_{i_1}, M\} > l_{i_1} \\ 0 & \text{otherwise.} \end{cases}$$

Let $i^1_\varphi(M) \equiv i_1$. For each $x_{i_1} \in [0, M]$, let $R^{x_{i_1}}_{i_1} \in \mathcal{R}$ be such that $p^{x_{i_1}}_{i_1} = x_{i_1}$ and $l^{x_{i_1}}_{i_1} = 0$. For each $\tilde{R}_{-i_1} \in \mathcal{R}^{N \setminus \{i_1\}}$, by Lemma 3, $\varphi_{i_1}(R^{x_{i_1}}_{i_1}, \tilde{R}_{-i_1}, M) = x_{i_1}$.

By *Pareto-efficiency* and *canonicity*, for each $\tilde{M} \in \mathbb{R}$ and each $\tilde{R}_{-i_1} \in \mathcal{R}^{N \setminus \{i_1\}}$, $\varphi_{i_1}(R^0_{i_1}, \tilde{R}_{-i_1}, \tilde{M}) = 0$.

Define the rule $\psi^{(M, i_1, x_{i_1})}$ for people in $N \setminus \{i_1\}$ by setting for each $\overline{M} \in \mathbb{R}$ and each $\overline{R}_{-i_1} \in \mathcal{R}^{N \setminus \{i_1\}}$,

$$\psi^{(M, i_1, x_{i_1})}(\overline{R}_{-i_1}, \overline{M}) = \begin{cases} \varphi_{-i_1}(R^{x_{i_1}}_{i_1}, \overline{R}_{-i_1}, M) & \text{if } \overline{M} = M - x_{i_1} \\ \varphi_{-i_1}(R^0_{i_1}, \overline{R}_{-i_1}, \overline{M}) & \text{otherwise.} \end{cases}$$

Because $\varphi_i(R^0_i, \overline{R}_{-i}, \overline{M}) = 0$ and φ is *Pareto-efficient, strategy-proof,* and *non-bossy*, $\psi^{(M, i_1, x_{i_1})}$ is well defined and inherits these properties. Thus, by the *induction hypothesis*, there is $I_{\psi^{(M, i_1, x_{i_1})}}$ such that $\psi^{(M, i_1, x_{i_1})} = CSP^{I_{\psi^{(M, i_1, x_{i_1})}}}$.

Let $i^2_\varphi(M, i_1, x_{i_1}) \equiv i^1_{\psi^{(M,i_1,x_{i_1})}}(M - x_{i_1}) = i_2$. For each $x_{i_2} \in [0, M - x_{i_1}]$, let $i^3_\varphi(M, i_1, i_2, x_{i_1}, x_{i_2}) \equiv i^2_{\psi^{(M,i_1,x_{i_1})}}(M - x_{i_1}, i_2, x_{i_2}) = i_3$. Proceeding this way, For each $x_{i_{n-2}} \in \left[0, M - \sum_{k=1}^{n-3} x_{i_k}\right]$, let

$$i^{n-1}_\varphi(M, i_1, \ldots, i_{n-2}, x_{i_1}, \ldots, x_{i_{n-2}}) \equiv i^{n-2}_{\psi^{(M,i_1,x_{i_1})}}(M - x_{i_1}, i_2, \ldots, i_{n-2}, x_{i_2}, \ldots, x_{i_{n-2}}).$$

Since we can do this for each $M \in \mathbb{R}$ and each $R_{i^1_\varphi(M)} \in \mathcal{R}$, we have a complete description of I_φ.

Step 2: Verification of $\varphi = CSP^{I_\varphi}$.

Let $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$. By definition of $i_1 \equiv i^1(M)$,

$$x_{i_1} \equiv \varphi_{i_1}(R, M) = \left\{ \begin{array}{ll} \min\{M, p_{i_1}\} & \text{if } \min\{M, p_{i_1}\} > l_{i_1} \\ 0 & \text{otherwise.} \end{array} \right\} = CSP^{I_\varphi}_{i_1}(R, M).$$

Further, by definition of $\psi^{(M,i_1,x_{i_1})}$,

$$\varphi_{-i_1}(R, M) = \psi^{(M,i_1,x_{i_1})}(R_{-i_1}, M - x_{i_1}) = CSP^{I_{\psi^{(M,i_1,x_{i_1})}}}(R_{-i_1}, M - x_{i_1}) = CSP^{I_\varphi}_{-i_1}(R, M).$$

Thus, $\varphi(R, M) = CSP^{I_\varphi}(R, M)$. Since this holds for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, we have verified that $\varphi = CSP^{I_\varphi}$. □

The axioms are independent. The *canonical equivalent* of the *uniform* rule violates only *Pareto-efficiency*. Certain *efficient uniform* rules violate only *strategy-proofness*. Next, we define a rule φ that violates only *non-bossiness*. Let $I \equiv \{i_k\}_{k=1}^n$ and $J \equiv \{j_k\}_{k=1}^n$ be such that $i_1 \neq j_1$ and $\{i_1, \ldots, i_{n-1}\} = \{j_1, \ldots, j_{n-1}\} = N \setminus \{1\}$. Define φ by setting, for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$,

$$\varphi(R, M) = \left\{ \begin{array}{ll} SP^I(R, M) & \text{if } l_1 < \min\{p_1, M\} \\ SP^J(R, M) & \text{otherwise.} \end{array} \right.$$

Notice that φ is also *weakly continuous*.

Remark 2. Strategy-proofness *and* Pareto-efficiency *are incompatible with equity.* Every *Pareto-efficient* and *canonical* rule is *non-bossy* whenever $|N| = 2$. Lemma 2 says that there is always one person who, independently of the preference profile, is satiated to the extent possible. It is in this sense that the combination of *strategy-proofness* and *Pareto-efficiency* precludes any meaningful notion of equity. ∘

We characterize the *weakly continuous* members of *conditional sequential priority* below.

Proposition 2. *A* conditional sequential priority rule *is* weakly continuous *if and only if it is a* sequential priority *rule.*

Proof: First we show that each *sequential priority* rule is *weakly continuous*. Let I be the ordering $i_1, i_2, \ldots, i_n$ of N . Let $\{(R^\nu, m^\nu)\}_{\nu=1}^\infty$ be a sequence in $\mathcal{R}^N \times \mathbb{R}_+$ such that,CSP

i) $\lim_{\nu\to\infty}(R^\nu, m^\nu) = (R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ and
ii) for each $i \in N$, if $l_i \geq \min\{p_i, M\}$, then there is $\nu^* \in \mathbb{N}$ such that for each $\nu \geq \nu^*, l^\nu_i \geq \min\{p^\nu_i, M^\nu\}$.

If $l_{i_1} \geq \min\{M, p_{i_1}\}$, then by ii), there is $\nu^* \in \mathbb{N}$ such that for each $\nu \geq \nu^*, l_{i_1}^{\nu} \geq \min\{p_{i_1}^{\nu}, M^{\nu}\}$, and so $SP_{i_1}^I(R^{\nu}, M^{\nu}) = 0$. Thus,

$$\lim_{\nu\to\infty} SP_{i_1}^I(R^{\nu}, M^{\nu}) = 0 = SP_{i_1}^I(R, M).$$

If $l_{i_1} < \min\{M, p_{i_1}\}$, then

$$\lim_{\nu\to\infty} SP_{i_1}^I(R^{\nu}, M^{\nu}) = \lim_{\nu\to\infty} \min\{p_{i_1}^{\nu}, M^{\nu}\} = \min\{p_{i_1}, M\} = SP_{i_1}^I(R, M).$$

This argument can be repeated for $i_2, \ldots, i_{n-1}$, and $i_n \in N \setminus \{i_1, \ldots, i_{n-1}\}$.

Next, we prove that if $I \equiv \{i^k\}_{k=1}^{n-2}$ is such that CSP^I is *weakly continuous*, then for each pair $M, M' \in \mathbb{R}_+$ and each pair $x, x' \in \mathbb{R}_+^N$ such that $\sum_{i=1}^{n-2} x_i \leq M$ and $\sum_{i=1}^{n-2} x_i' \leq M$,

$$\begin{aligned}
i^1(M) \quad &= i_1 = \quad i^1(M'),\\
i^2(M, i_1, x_{i_1}) \quad &= i_2 = \quad i^2(M', i_1, x'_{i_1}),\\
&\ \ \vdots\\
i^{n-1}(M, i_1, \ldots, i_{n-2}, x_{i_1}, \ldots, x_{i_{n-2}}) \quad &= i_{n-1} = \quad i^{n-1}(M', i_1, \ldots, i_{n-2}, x_{i_1}, \ldots, x'_{i_{n-2}}),
\end{aligned}$$

Let $R \in \mathcal{R}^N$ be such that for each $i \in N, l_i = 0$ and $p_i = \max\{M, M'\}$. By definition of CSP^I and R, for each $\tilde{M} \leq \max\{M, M'\}, CSP^I(R, \tilde{M}) \in \{(\tilde{M}, 0, \ldots, 0), \ldots, (0, \ldots, 0, \tilde{M})\}$. Since for each $i \in N$, $l_i = 0$, *weak continuity* implies that $CSP^I(R, \tilde{M})$ varies continuously in $\tilde{M}$. Thus, $i^1(M) = i^1(M') = i_1$.

For each $\tilde{M} \in \mathbb{R}_+$ and each $\tilde{x}_{i_1} \in \mathbb{R}_+$ such that $\tilde{x}_{i_1} \leq \tilde{M}$, let $R_{i_1}^{\tilde{x}_{i_1}} \in \mathcal{R}$ be such that $l_{i_1}^{\tilde{x}_{i_1}} = 0$ and $p_{i_1}^{\tilde{x}_{i_1}} = \tilde{x}_{i_1}$. Then $CSP_{i_1}^I(R_{i_1}^{\tilde{x}}, R_{-i^1}, \tilde{M}) = \tilde{x}_{i_1}$ and $CSP_{-i_1}^I(R_{i_1}^{\tilde{x}_{i_1}}, R_{-i_1}, \tilde{M}) \in \{(\tilde{M} - \tilde{x}_{i_1}, 0, \ldots, 0), \ldots, (0, \ldots, 0, \tilde{M} - \tilde{x}_{i_1})\}$. Since for each $i \in N \setminus \{i_1\}$, $l_i = 0$, *weak continuity* implies that $CSP_{-i_1}^I(R_{i_1}^{\tilde{x}_{i_1}}, R_{-i_1}, \tilde{M})$ varies continuously with $\tilde{M}$ and $\tilde{x}_{i_1}$. Thus, $i^2(M, i_1, x_{i_1}) = i^2(M', i_1, x'_{i_1}) = i_2$.

Repeating this argument completes the proof. □

The following is a corollary of Proposition 2 and Theorem 2.

Corollary 2. *A rule is* Pareto-efficient, strategy-proof, non-bossy *and* weakly continuous *if and only if it is a* sequential priority rule.

The axioms are independent. The *canonical equivalent* of the *uniform* rule violates only *Pareto-efficiency*. There are *efficient uniform* rules that violate only *strategy-proofness*. The rule φ defined to show independence of the axioms in Theorem 2 violates only *non-bossiness*. Finally, each *conditional sequential priority rule* that is not an (unconditional) *sequential priority* rule violates only *weak continuity*.

We end this section with a list of incompatibilities.

Proposition 3. *Each of the following sets of axioms is incompatible:*

1. Pareto-efficiency *and* equal treatment of equals.
2. Pareto-efficiency *and* envy-freeness.
3. Pareto-efficiency *and* continuity.

4. Pareto-efficiency, strategy-proofness, *and* awardee-envy-freeness.
5. Pareto-efficiency, strategy-proofness, *and the* equal-division lower bound.
6. Pareto-efficiency, strategy-proofness *and* welfare-dominance under preference replacement.
7. Group strategy-proofness *and* unanimity.

Proof:[14] Let φ be *Pareto-efficient*. Let $(R,M) \in \mathcal{R}^N \times \mathbb{R}_+$ be such that for each $i \in N$, $l_i = \frac{M}{n}$ and $p_i = M$. By *Pareto-efficiency*, there is $i \in N$ such that $\varphi_i(R,M) > l_i = \frac{M}{n}$. This means that there is $j \in N \setminus \{i\}$ such that $\varphi_j(R,M) < \frac{M}{n}$. Thus, φ violates *equal treatment of equals*, and establishes incompatibility 1. Since *envy-freeness* implies *equal treatment of equals*, we also have incompatibility 2.

Let $R \in \mathcal{R}^N$ be such that for each $i \in N$, $p_i = M$ and $l_i = 0$. By *Pareto-efficiency*, there is $i \in N$ such that $\varphi_i(R,M) > l_i$. Define $r_i : [0,1] \to \mathcal{R}$ by setting, for each $t \in [0,1]$, $r_i(t) = R_i^t \in \mathcal{R}$ associated with $p_i^t = M$ and $l_i^t = tM$. Then, $r_i(0) = R_i$ and $r_i(1)$ is associated with $l_i^1 = M$. Define $\gamma : [0,1] \to [0,M]$ by setting, for each $t \in [0,1], \gamma(t) \equiv \varphi_i(r_i(t), R_{-i}, M)$. Then, $\gamma(0) > 0$. Since $r_i(1)$ is associated with $l_i^t = M$, and for each $j \in N \setminus \{i\}$, $l_j = 0$ and $p_j = M$, by *Pareto-efficiency*, $\gamma(1) = \varphi_i(r_i(1), R_{-i}, M) = 0$. Further, by *Pareto-efficiency* since for each $j \in N \setminus \{i\}$, $l_j = 0$ and $p_j = M$, there is no $t \in [0,1]$ such that $\gamma(t) \in (0, tM]$. So γ, and thus φ, is *discontinuous*. This establishes incompatibility 3.

We show incompatibilities 4. and 5. for $|N| = 2$. The arguments generalize easily to $|N| > 2$. Suppose that φ is *strategy-proof* in addition to being *Pareto-efficient*. Let $N \equiv \{1,2\}$ and $M \equiv 6$.

Suppose that φ is *awardee-envy-free*. Let $R \in \mathcal{R}^N$ be such that $l_1 = l_2 = 0$ and $p_1 = p_2 = 6$. By *Pareto-efficiency* and *awardee-envy-freeness*, $\varphi(R,M) \in \{(0,6),(3,3),(6,0)\}$. If $\varphi(R,M) = (0,6)$, then let $R_2' \in \mathcal{R}$ be such that $l_2' = 0$, $p_2' = 5$. By *strategy-proofness*, $\varphi_2(R_1, R_2', M) \geq 5$: otherwise, $6 = \varphi_2(R_1, \overset{\mathrm{F}}{R_2}, M) \overset{\mathrm{T}}{P_2'} \varphi_2(R_1, \overset{\mathrm{T}}{R_2'}, M)$. By *Pareto-efficiency*, $\varphi(R_1, R_2', M) = (1,5)$. But this violates *awardee-envy-freeness* at (R_1, R_2'). By an analogous argument, $\varphi(R,M) \neq (6,0)$. Thus, $\varphi(R,M) = (3,3)$. Now let $\overline{R}_1 \in \mathcal{R}$ be such that $\overline{l}_1 = 2$ and $\overline{p}_1 = 5$. By *strategy-proofness*, $\varphi_1(\overline{R}_1, R_2, M) = 3$. Let $\overline{R}_2 = \overline{R}_1$. Again, by *strategy-proofness*, $\varphi(\overline{R}, M) = (3,3)$. Let $\tilde{R}_1 \in \mathcal{R}$ such that $\tilde{l}_1 = 4$ and $\tilde{p}_1 = 5$. By *strategy-proofness*, $\varphi_1(\tilde{R}_1, \overline{R}_2, M) \leq 3$. By *Pareto-efficiency*, $\varphi_2(\tilde{R}_1, \overline{R}_2, M) \geq 5$. By an analogous argument, letting $\tilde{R}_2 = \tilde{R}_1$, $\varphi_1(\overline{R}_1, \tilde{R}_2, M) \geq 5$. Finally, by *Pareto-efficiency*, either $\varphi_1(\tilde{R}, M) \geq 5$ or $\varphi_2(\tilde{R}, M) \geq 5$. However, if $\varphi_1(\tilde{R}, M) \geq 5$, then $\varphi_2(\tilde{R}_1, \overset{\mathrm{F}}{\overline{R}_2}, M) \overset{\mathrm{T}}{\tilde{P}_2} \varphi_2(\tilde{R}_1, \overset{\mathrm{T}}{\tilde{R}_2}, M)$. Thus, $\varphi_2(\tilde{R}, M) \geq 5$. Since $\varphi_1(\overset{\mathrm{F}}{\overline{R}_1}, \tilde{R}_2, M) \overset{\mathrm{T}}{\tilde{P}_1} \varphi_2(\overset{\mathrm{T}}{\tilde{R}_1}, \tilde{R}_2, M)$, this violates *strategy-proofness* and establishes incompatibility 4.

Suppose that φ meets the *equal-division lower bound*. Let $R \in \mathcal{R}^N$ be such that $l_1 = l_2 = 3$ and $p_1 = p_2 = 6$. By *Pareto-efficiency*, $\varphi(R,M) \in \{(6,0),(0,6)\}$. If $\varphi(R,M) = (0,6)$, then let $R_1' \in \mathcal{R}$ be such that $l_1' = 0$ and $p_1' = 6$. By the *equal-division lower bound*, $\varphi_1(R_1', R_2, M) \geq 3$ and so $\varphi_2(R_1', R_2, M) \leq 3$. By *Pareto-efficiency*, $\varphi(R_1', R_2, M) = (6,0)$. Then, $6 = \varphi_1(\overset{\mathrm{F}}{R_1'}, R_2, M) \overset{\mathrm{T}}{P_i} \varphi_1(\overset{\mathrm{T}}{R_1}, R_2, M) = 0$. Thus, φ violates

[14] The proofs of 6. and 7. are available from the author upon request.

strategy-proofness. The case where $\varphi(R, M) = (6, 0)$ is analogous. This establishes incompatibility 5. □

6. Conclusion

Equal treatment of equals, and therefore *envy-freeness*, is incompatible with *Pareto-efficiency*. Yet, other axioms such as *awardee envy-freeness* and the *equal-division lower bound* are compatible with it. However, none of these can be imposed together with *Pareto-efficiency* and *strategy-proofness*. The implication of Theorem 2 is that if we insist on both *Pareto-efficiency* and *strategy-proofness*, then meaningful concepts of fairness are untenable. If we give up on *strategy-proofness*, Corollary 1 is a characterization of all *Pareto-efficient* and *awardee-envy-free* rules that satisfy the *equal-division lower bound*.

The results presented here are reminiscent of those found in the literature on problems with discrete goods such as the allocation of objects. This is particularly true of Theorem 2 (Pápai, 2001; Ehlers and Klaus, 2003; Hatfield, 2009). In discrete environments, a way to recover equity is by randomization (Hofstee, 1990). We leave the study of randomized rules for future work.

It is easy to extend the definitions of rules presented here to the natural variable-population environment. In addition to the axioms presented in the main text, some interesting axioms in such an environment are "consistency" (Balinski and Young, 2001; Thomson, 1988), "population monotonicity" (Thomson, 1983a,b), and "replication invariance" (Thomson, 1997). Following are a few results that can be shown:[15]

1. *A* conditional sequential priority *rule is* consistent *if and only if it is a* sequential priority *rule.*
2. *No* conditional sequential priority *rule is* population monotonic*.*
3. Pareto-efficiency *and* replication invariance *are incompatible.*

7. References

Balinski, Michel L and H Peyton Young (2001) *Fair Representation: Meeting the Ideal of One Man, One Vote*: Brookings Institution Press.

Bénassy, Jean-Pascal (1982) *The economics of market disequilibrium.*

Bergantiños, Gustavo, Jordi Massó, and Alejandro Neme (2009) "The Division Problem with Voluntary Participation," *MIMEO.*

——— (2010) "The division problem with maximal capacity constraints," working paper.

Ching, Stephen (1992) "A simple characterization of the uniform rule," *Economics Letters*, Vol. 40, No. 1, pp. 57–60, Sep.

——— (1994) "An alternative characterization of the uniform rule," *Social Choice and Welfare*, Vol. 11, pp. 131–136.

Ehlers, Lars and Bettina Klaus (2003) "Coalitional strategy-proof and resource-monotonic solutions for multiple assignment problems," *Social Choice and Welfare*, Vol. 21, No. 2, pp. 265–280, Oct.

Hatfield, John (2009) "Strategy-proof, efficient, and nonbossy quota allocations," *Social Choice and Welfare*, Vol. 33, No. 3, pp. 505–515, September.

Hofstee, Willem (1990) "Allocation by lot: A conceptual and empirical analysis," *Social Science Information*, Vol. 39, No. 4, pp. 745–763.

Kıbrıs, Özgür (2003) "Constrained allocation problems with single-peaked preferences: An axiomatic analysis," *Social Choice and Welfare*, Vol. 20, No. 3, pp. 353–362, 06.

[15]Proofs of these results are available from the author upon request.

Moulin, Hervé (1987) "The Pure Compensation Problem: Egalitarianism versus Laissez-Fairism," *The Quarterly Journal of Economics*, Vol. 102, No. 4, pp. 769–83, Nov.

O'Neill, Barry (1982) "A problem of rights arbitration from the Talmud," *Mathematical Social Sciences*, Vol. 2, No. 4, pp. 345–371, Jun.

Pápai, Szilvia (2001) "Strategyproof and Nonbossy Multiple Assignments," *Journal of Public Economic Theory*, Vol. 3, No. 3, pp. 257–271.

Satterthwaite, Mark A and Hugo Sonnenschein (1981) "Strategy-Proof Allocation Mechanisms at Differentiable Points," *Review of Economic Studies*, Vol. 48, No. 4, pp. 587–97, Oct.

Schummer, James and William Thomson (1997) "Two derivations of the uniform rule and an application to bankruptcy," *Economics Letters*, Vol. 55, No. 3, pp. 333–337, Sep.

Sönmez, Tayfun (1994) "Consistency, monotonicity, and the uniform rule," *Economics Letters*, Vol. 46, No. 3, pp. 229–235, Nov.

Sprumont, Yves (1991) "The division problem with single-peaked preferences: A characterization of the uniform rule," *Econometrica*, Vol. 59, No. 2, pp. 509–519, Mar.

Thomson, William (1983a) "The Fair Division of a Fixed Supply Among a Growing Population," *Mathematics of Operations Research*, Vol. 8, No. 3, pp. 319–326, Aug.

——— (1983b) "Problems of fair division and the Egalitarian solution," *Journal of Economic Theory*, Vol. 31, No. 2, pp. 211–226, Dec.

——— (1988) "A study of choice correspondences in economies with a variable number of agents," *Journal of Economic Theory*, Vol. 46, No. 2, pp. 237–254, Dec.

——— (1994) "Consistent Solutions to the Problem of Fair Division When Preferences Are Single-Peaked," *Journal of Economic Theory*, Vol. 63, No. 2, pp. 219–245, Aug.

——— (1995) "Population-Monotonic Solutions to the Problem of Fair Division When Preferences Are Single-Peaked," *Economic Theory*, Vol. 5, No. 2, pp. 229–46, Mar.

——— (1997) "The Replacement Principle in Economies with Single-peaked Preferences," *Journal of Economic Theory*, Vol. 76, No. 1, pp. 145–168.

——— (1999) "Welfare-domination under preference-replacement: A survey and open questions," *Social Choice and Welfare*, Vol. 16, No. 3, pp. 373–394.

Weymark, John A (1999) "Sprumont's characterization of the uniform rule when all single-peaked preferences are admissible," *Review of Economic Design*, Vol. 4, No. 4, pp. 389–393.