Implementation in Stochastic Dominance Nash Equilibria*

Eun Jeong Heo[†] and Vikram Manjunath[‡]

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Abstract

We study solutions that choose lotteries for profiles of preferences defined over sure alternatives. We define Nash equilibria based on "stochastic dominance" comparisons and study the implementability of solutions in such equilibria. We show that a Maskin-style invariance condition is necessary and sufficient for implementability. Our results apply to an abstract Arrovian environment as well as a broad class of economic environments.

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1 Introduction

The problem of selecting lotteries over alternatives while taking as input only ordinal preferences over sure alternatives dates back to Gibbard (1977). An important feature of such problems is that preferences are not defined over the entire set from which the collective choice is made. We are interested in the "implementability" of a solution for this problem: there ought to be a "game form" such that for each profile of agents' preferences, the equilibrium outcomes of the induced game are exactly what the solution would have chosen for that preference profile. For this to be precise, we need to first specify an equilibrium concept. That outcomes are lotteries but ordinal preferences are only defined over sure alternatives means that we cannot speak of the usual Nash equilibria. This is the starting point of our study.

Suppose that the primitives of the model only specify an ordinal ranking of the sure alternatives. It is very reasonable to suppose that if one lottery first order stochastically dominates the other,[1] then it is preferred to the other. This is an extension of the primitive ordinal ranking over sure alternatives to lotteries. As the first-order stochastic dominance relation is *incomplete*, there are two ways we can define Nash equilibria: A profile of strategies such that for each agent, there is no strategy to which he can deviate in order to achieve an outcome that

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[†]Department of Economics at Vanderbilt University (heoeunjeong@gmail.com)

[‡]Department of Economics at Texas A&M University, (vmanjunath@tamu.edu)

[1]By this we mean that probability of each of the following events is at least as likely (and at least one is more likely) under the first lottery as it is under the second: the top alternative is realized, one of the top two alternatives is realized, one of the top three alternatives is realized, and so on.

1. stochastically dominates the outcome when he does not deviate.

2. is neither the same as nor stochastically dominated by the outcome when he does not deviate.

It is straightforward that every equilibrium of the second kind is an equilibrium of the first kind (Proposition 1). So we call the first a "weak stochastic dominance Nash equilibrium" (weak SDNE) and the second a "strong stochastic dominance Nash equilibrium" (strong SDNE).

Equilibrium notions similar to strong SDNE have previously appeared in the implementation literature (Bochet 2007, Benoît and Ok 2008). Ehlers and Masso (2007) and Pais (2008) define similar equilibrium notions in the analysis of matching markets. *Ordinal Bayesian incentive compatibility* (Majumdar and Sen 2004) is also based on this type of comparison between lotteries. The notion of weak SDNE, however, has only appeared more recently. In the context of probabilistic allocation, Ekici and Kesten (2010) study the weak SDNE of a particular game: the preference revelation game induced by the *Bogomolnaia-Moulin* solution.[2]

We are interested in solutions that only take ordinal preferences over *sure* alternatives as input. However, if the primitive preferences are cardinal, despite being defined only over sure alternatives, they might tell us something about how an agent would compare lotteries. For instance, if the primitives were Bernoulli utility functions, an agent could compare lotteries based on expected utility. The relevant notion of Nash equilibrium would be the standard one: a profile of strategies such that for each agent, there is no strategy to which he can deviate to achieve an outcome that yields a higher expected utility than the outcome when he does not deviate. We call these "expected utility Nash equilibria."

We answer a natural question: Which solutions can be implemented in each of these three kinds of equilibria? Maskin's "monotonicity" condition (Maskin 1999) is central to the literature on implementation. We adapt it to our setting and define three separate "invariance" conditions,[3] each of which is necessary and sufficient for implementability in a different kind of equilibrium (Proposition 2). Does this mean that there are solutions that can be implemented in one kind of equilibrium but not another? Our main result is the answer to this question: a solution is implementable in one kind of equilibrium if and only if it is implementable in each of the other kinds of equilibria (Theorem 1). To prove this, we show that each of the three aforementioned invariance conditions is equivalent to a fourth condition that we call "invariance to support monotonic transformations" (Proposition 3).

It is unsurprising that implementability is not a terribly demanding requirement in our setting. An insight from the literature on "virtual implementation" (Matsushima 1988, Abreu and Sen 1991) has some bearing here: At lotteries that have full support, there are no Maskin monotonic transformations of a *von Neumann-Morgenstern* preference. Consequently, among solutions that select lotteries of full support, implementability is a vacuous property. There are a few closely related settings where invariance to Maskin-monotonic transformations is necessary and sufficient for implementability. In Bochet and Sakai (2005) a solution maps each profile of preferences over lotteries to a set of lotteries. In both Bochet (2007) and Benoît and Ok (2008) a solution maps each profile of preferences over sure alternatives to a set of sure alternatives.[4]

[2] This is often referred to as the "probabilistic serial rule."

[3] We have adopted the terminology of Thomson (2013).

[4] Note that in these two papers, a game form used to implement such solutions allows randomization over sure alternatives and strong SDNE is adopted as the equilibrium concept. However, as long as the game form *implements* the solution selecting sure alternatives, equilibrium outcomes are sure alternatives. That is, lotteries only appear off the equilibrium path.

In each case, the preference profile that a solution takes as input is defined over its range. So the standard definition of Maskin-invariance applies. In our setting, however, a solution maps each profile of preferences over *sure* alternatives to a set of *lotteries*. For this reason, we need an adaptation of Maskin's condition.

We show our results in an abstract Arrovian setting where we assume that preferences over sure alternatives are strict. However, subject to a mild regularity condition on solutions, they extend to a broad class of economic environments. This class includes the probabilistic assignment of single objects (Hylland and Zeckhauser 1979, Bogomolnaia and Moulin 2001), multiple objects (Kojima 2009, Kasajima 2011, Heo 2014), and packages (Budish 2011, Budish, Che, Kojima and Milgrom 2013, Peivandi 2014). It is important to note that, unlike in some economic environments including classical exchange, "no veto" is not a vacuous property in the economic environments that we study.

Finally, as a concrete example of an economic environment, we study the probabilistic assignment of single objects. We check if some prominent solutions discussed in the literature are implementable. Our results stand in sharp contrast to the existing impossibility results involving "strategy-proofness," which requires that it be a weakly dominant strategy to report preferences truthfully in the stochastic dominance sense. First, the Pareto solution, the no-envy solution, and their intersection are all implementable while there is no *strategy-proof* selection from the intersection (Bogomolnaia and Moulin 2001). Second, the Pareto solution, the *endowment lower bound* solution, and their intersection are implementable while there is no *strategy-proof* selection from the intersection (Athanassoglou and Sethuraman 2011). We also provide a characterization (for the three-person case) of the *Bogomolnaia-Moulin* solution as the only implementable selection from the intersection of the Pareto and no-envy solutions.

The remainder of this paper is organized as follows. In Section 2 we define an abstract Arrovian environment: we first describe the primitives of the model (Subsection 2.1) and then define game forms and the three equilibrium concepts (Subsection 2.2). In Section 3 we define three invariance conditions. In Section 4 we show that each of these conditions is necessary and sufficient for implementability in one the equilibrium concepts. We then show that they are, in fact, equivalent. In Section 5 we define a class of economic environments to which our results readily extend. In Section 6 we study the implications of implementability for the probabilistic assignment problem of single objects. Many of the proofs are in the Appendix.

2 An abstract Arrovian environment

2.1 Primitives of the model

Let A be a finite set of alternatives and N be a set of agents where $|A| \geq 3$ and $|N| \geq 3$. Each $i \in N$ ranks the alternatives in A according to a linear order P_i. For each pair $a, b \in A$, we write $a\ P_i\ b$ if i prefers a to b. We write $a\ R_i\ b$ if either $a\ P_i\ b$ or $a = b$. Let $\boldsymbol{\mathcal{P}}$ be the set of such linear orders over A. A **preference profile** is a list $P \equiv (P_i)_{i \in N} \in \mathcal{P}^N$. For each $P_0 \in \mathcal{P}$ and each $a \in A$, let the upper contour set of P_0 at a be $\boldsymbol{U(P_0, a)} \equiv \{b \in A : b\ R_0\ a\}$. Similarly, let the lower contour set of P_0 at a be $\boldsymbol{L(P_0, a)} \equiv \{b \in A : a\ R_0\ b\}$.

Let $u_0 : A \to \mathbb{R}$ be a Bernoulli utility function for which there is no pair $a, b \in A$ such that $a \neq b$ yet $u_0(a) = u_0(b)$. Let $\boldsymbol{\mathcal{U}}$ be the set of such Bernoulli utility functions. For each $P_0 \in \mathcal{P}$, we say that $\boldsymbol{u_0}$ **is consistent with** $\boldsymbol{P_0}$ if for each pair $a, b \in N$, $a\ P_0\ b$ if and only if $u_0(a) > u_0(b)$. The set of Bernoulli utility functions consistent with P_0 is $\boldsymbol{\mathcal{U}(P_0)}$. A profile of Bernoulli utility functions is a list $u \equiv (u_i)_{i \in N} \in \mathcal{U}^N$. For each $P \in \mathcal{P}^N$, let $\boldsymbol{\mathcal{U}(P)} \equiv \mathcal{U}(P_1) \times \cdots \times \mathcal{U}(P_n)$.

A **lottery** is a list $\pi \equiv (\pi(a))_{a\in A} \in \mathbb{R}_+^{|A|}$ such that $\sum_{a\in A}\pi(a) = 1$. For each $a \in A$, $\pi(a)$ represents the probability that a is realized. For each $B \subseteq A$, let $\pi(B) \equiv \sum_{b\in B}\pi(b)$. Let $\boldsymbol{\Delta(A)}$ be the set of all lotteries.

For each $\pi \in \Delta(A)$ and each $u_0 \in \mathcal{U}$, let $\boldsymbol{U_0(\pi)} \equiv \sum_{a\in A}\pi(a)u_0(a)$ be the expected utility from π under Bernoulli utility function u_0.

A **solution** is a nonempty-valued correspondence $\varphi : \mathcal{P}^N \rightrightarrows \Delta(A)$.

Let $P_0 \in \mathcal{P}$ and π, $\pi' \in \Delta(A)$. Since P_0 is a strict preference relation over sure alternatives in A, not over $\Delta(A)$, one cannot compare π and π' using just P_0. We consider three ways of comparing lotteries based on either stochastic dominance (SD) or expected utility. We say that **$\boldsymbol{\pi}$ stochastically dominates $\boldsymbol{\pi'}$ at $\boldsymbol{P_0}$**—we denote this as $\boldsymbol{\pi\ P_0^{sd}\ \pi'}$—if for each $a \in A$, $\pi(U(P_0, a)) \geq \pi'(U(P_0, a))$ with at least one strict inequality.[5] We write $\boldsymbol{\pi\ R_0^{sd}\ \pi'}$ if either $\pi\ P_0^{sd}\ \pi'$ or $\pi = \pi'$. Note that R_0^{sd} is not a complete binary relation over $\Delta(A)$.[6] For each $\pi \in \Delta(A)$, the **SD upper contour set of $\boldsymbol{P_0}$ at $\boldsymbol{\pi}$** is $\boldsymbol{U^{sd}(P_0, \pi)} \equiv \{\pi' \in \Delta(A) : \pi'\ R_0^{sd}\ \pi\}$. The **SD lower contour set of $\boldsymbol{P_0}$ at $\boldsymbol{\pi}$** is $\boldsymbol{L^{sd}(P_0, \pi)} \equiv \{\pi' \in \Delta(A) : \pi\ R_0^{sd}\ \pi'\}$. For each $u \in \mathcal{U}$, the **expected utility lower contour set of $\boldsymbol{u_0}$ at $\boldsymbol{\pi}$** is $\boldsymbol{L^{eu}(u_0, \pi)} \equiv \{\pi' \in \Delta(A) : U_0(\pi') \leq U_0(\pi)\}$. Each of $L^{sd}(P_0, \pi)$, $U^{sd}(P_0, \pi)$, and $L^{eu}(u_0, \pi)$ is a closed and convex set.

2.2 Game forms and equilibria

A **game form** consists of, for each $i \in N$, a **strategy space**, $\boldsymbol{S_i}$, and an **outcome function**, $\boldsymbol{h}$, which associates each profile of strategies with an outcome in $\Delta(A)$. We denote a generic game form by $\Gamma = (S, h)$ with $S \equiv S_1 \times \cdots \times S_n$ and $h : S \to \Delta(A)$. For each $i \in N$ and each $s \in S$, we denote the strategy of everyone but i by $s_{-i} \equiv (s_j)_{j\in N\setminus\{i\}}$.

It is clear that if an agent could achieve, by deviating, a lottery that he finds better in the SD sense he would do so. Similarly, if an agent could only achieve, by deviating, a lottery that he finds worse in the SD sense he would not do so. What should he do when deviating yields a lottery that is not comparable? There are three possibilities.

(1) **Strong SD Nash equilibrium**: An agent deviates if doing so yields a lottery that is neither stochastically dominated by nor equal to what he receives when he does not deviate. Let $P \in \mathcal{P}^N$ and $s \in S$. We say that s is a **strong SDNE of $\boldsymbol{(\Gamma, P)}$** if for each $i \in N$ and each $s'_i \in S_i$, $h(s_i, s_{-i})\ R_i^{sd}\ h(s'_i, s_{-i})$. Let $\boldsymbol{SNE^{sd}(\Gamma, P)}$ be the set of all such equilibria.

(2) **Weak SD Nash equilibrium**: An agent deviates if doing so yields a lottery that stochastically dominates what he receives when he does not deviate. Let $P \in \mathcal{P}^N$, and $s \in S$. We say that s is a **weak SDNE of $\boldsymbol{(\Gamma, P)}$** if for each $i \in N$, there is no $s'_i \in S_i$ such that $h(s'_i, s_{-i})\ P_i^{sd}\ h(s_i, s_{-i})$. Let $\boldsymbol{WNE^{sd}(\Gamma, P)}$ be the set of all such equilibria.

(3) **Expected utility Nash equilibrium**: This is the usual notion of a Nash equilibrium for given Bernoulli utility functions. Let $u \in \mathcal{U}^N$ and $s \in S$. We say that s is an **expected utility NE of $\boldsymbol{(\Gamma, u)}$** if for each $i \in N, s_i \in \text{argmax}_{s'_i\in S_i} U_i(h(s'_i, s_{-i}))$. Let $\boldsymbol{NE(\Gamma, u)}$ be the set of all such equilibria.

[5] This is equivalent to stating that for each $a \in A$, $\pi'(L(P_0, a)) \geq \pi(L(P_0, a))$ with at least one strict inequality and equality for the best alternative in A according to P_0.

[6] One way to complete this relation is to treat non-comparability as indifference. The resulting complete relation, however, will not be transitive. For example, consider P_0 such that $a\ P_0\ b\ P_0\ c$ and three lotteries $\pi_0 = (0.4, 0, 0.6)$, $\pi'_0 = (0, 1, 0)$, and $\pi''_0 = (0.6, 0, 0.4)$. Neither π_0 and π'_0 nor π'_0 and π''_0 are comparable. However, $\pi''_0\ P_0^{sd}\ \pi_0$.

While *strong SDNE* and *expected utility NE* have been studied previously, *weak SDNE* is new in the implementation context. For each game form, each profile of ordinal preferences, and each profile of Bernoulli utilities consistent with the ordinal profile, every strong SDNE is an expected utility NE and every expected utility NE is a weak SDNE.

Proposition 1. For each Γ, each $P \in \mathcal{P}^N$, and each $u \in \mathcal{U}(P)$,

$$SNE^{sd}(\Gamma, P) \subseteq NE(\Gamma, u) \subseteq WNE^{sd}(\Gamma, P).$$

Proof. Let $s \in SNE^{sd}(\Gamma, P)$ and $i \in N$. Then, for each $s_i' \in S_i$, $h(s_i, s_{-i})\ R_i^{sd}\ h(s_i', s_{-i})$. If $h(s_i, s_{-i}) = h(s_i', s_{-i})$, then for each $u_i \in \mathcal{U}(P_i)$, $U_i(h(s_i, s_{-i})) = U_i(h(s_i', s_{-i}))$. If $h(s_i, s_{-i})\ P_i^{sd}\ h(s_i', s_{-i})$, then for each $u_i \in \mathcal{U}(P_i)$, $U_i(h(s_i, s_{-i})) > U_i(h(s_i', s_{-i}))$. Thus, $s_i \in \text{argmax}_{s_i' \in S_i} U_i(h(s_i', s_{-i}))$ and so $s \in NE(\Gamma, u)$. This establishes that $SNE^{sd}(\Gamma, P) \subseteq NE(\Gamma, u)$.

Suppose $s \notin WNE^{sd}(\Gamma, P)$. Then there are $i \in N$ and $s_i' \in S_i$ such that $h(s_i', s_{-i})\ P_i^{sd}\ h(s_i, s_{-i})$. However, this implies that for each $u_i \in \mathcal{U}(P_i)$, $U_i(h(s_i', s_{-i})) > U_i(h(s_i, s_{-i}))$. Thus, $s \notin NE(\Gamma, u)$. This establishes that $NE(\Gamma, u) \subseteq WNE^{sd}(\Gamma, P)$. □

Remark 1. There are games where the inclusions of Proposition 1 are strict. We present an example in Appendix A.

We next define implementability of a solution. Let φ be a solution and $\Gamma = (S, h)$ be a game form. For each $T \subseteq S$, let $h(T) \equiv \bigcup_{s \in T} \{h(s)\}$. We say that **$\Gamma$ implements φ in strong SDNE** (or **weak SDNE**) if for each $P \in \mathcal{P}^N$, $h(SNE^{sd}(\Gamma, P)) = \varphi(P)$ (or $h(WNE^{sd}(\Gamma, P)) = \varphi(P)$). That is, for each profile of preferences, the set of strong SDNE (or weak SDNE) outcomes corresponds to the set of lotteries that the solution recommends.

Recall that we consider solutions that only take agents' ordinal preferences as input. Let $\rho : \mathcal{U} \to \mathcal{P}$ map each Bernoulli utility function to the ordinal preference that it is consistent with.[7] For each $u \in \mathcal{U}^N$, let $\rho(u) \equiv (\rho(u_i))_{i \in N}$. Then, $\varphi \circ \rho$ maps profiles of Bernoulli utility functions to lotteries. We say that **Γ implements φ in expected utility NE** if for each $u \in \mathcal{U}^N$, $h(NE(\Gamma, u)) = \varphi \circ \rho(u)$. That is, for each profile of Bernoulli utility functions, the set of expected utility NE outcomes corresponds to the set of lotteries that the solution recommends.

A solution is **implementable in strong SDNE** (or **weak SDNE** or **expected utility NE**) if there is a *game form* that implements it in *strong SDNE* (or *weak SDNE* or *expected utility NE*).

Remark 2. These notions of implementability are ones of *complete information.* As with the literature that follows Maskin (1999), we interpret the results in this paper as an exploration of the logical implications of the complete information assumption. For analyses of incomplete information settings we refer the reader to the literature on Bayesian mechanism design (d'Aspremont and Gérard-Varet 1979, Dasgupta, Hammond and Maskin 1979, Myerson 1979, Harris and Townsend 1981).

3 Invariance conditions

In this section, we present three "invariance" conditions on solutions, each being associated with a certain kind of change in preferences.[8] We first define the three types of changes.

[7] That is, for each $u_0 \in \mathcal{U}$, $u_0 \in \mathcal{U}(\rho(u_0))$.

[8] We adopt the terminology of Thomson (2013). By "Maskin-invariance" we mean the property known in the literature as "Maskin-monotonicity." Thus, we refer to the new conditions that we introduce as "invariance" conditions rather than as "monotonicity" conditions.

For each $P_0, P_0' \in \mathcal{P}$ and each $\pi \in \Delta(A)$,

(1) $\boldsymbol{P_0'}$ is a **lower SD monotonic transformation of $\boldsymbol{P_0}$ at $\boldsymbol{\pi}$** if $L^{sd}(P_0, \pi) \subseteq L^{sd}(P_0', \pi)$.

(2) $\boldsymbol{P_0'}$ is an **upper SD monotonic transformation of $\boldsymbol{P_0}$ at $\boldsymbol{\pi}$** if $U^{sd}(P_0', \pi) \subseteq U^{sd}(P_0, \pi)$.

For each $u_0, u_0' \in \mathcal{U}$ and each $\pi \in \Delta(A)$,

(3) $\boldsymbol{u_0'}$ is a **Maskin monotonic transformation of $\boldsymbol{u_0}$ at $\boldsymbol{\pi}$** if $L^{eu}(u_0, \pi) \subseteq L^{eu}(u_0', \pi)$.

For each $P, P' \in \mathcal{P}^N$ and each $\pi \in \Delta(A)$, we say that P' is a *lower SD (upper SD) monotonic transformation of P at π* if, for each $i \in N$, P_i' is a *lower SD (upper SD) monotonic transformation of P_i at π*. Similarly, for each $u, u' \in \mathcal{U}^N$ and each $\pi \in \Delta(A)$, we say that u' is a *Maskin monotonic transformation of u at π* if, for each $i \in N$, u_i' is a *Maskin monotonic transformation of u_i at π*.

Now, we define the corresponding invariance conditions. Let φ be a solution.

Invariance to lower (upper) SD monotonic transformations: For each $P \in \mathcal{P}^N$, each $\pi \in \varphi(P)$, and each $P' \in \mathcal{P}^N$, if P' is a *lower (upper) SD monotonic transformation of P at π*, then $\pi \in \varphi(P')$.

Invariance to Maskin monotonic transformations: For each $u \in \mathcal{U}^N$, each $\pi \in \varphi \circ \rho(u)$, and each $u' \in \mathcal{U}^N$, if u' is a *Maskin monotonic transformation of u at π*, then $\pi \in \varphi \circ \rho(u')$.

4 Necessary and sufficient conditions for implementability

In various contexts where randomization in the game form is permitted, a Maskin-style invariance condition turns out to be a necessary and sufficient for implementability (Bochet 2007, Benoît and Ok 2008, Bochet and Sakai 2005). We show this in the current context for both *strong* and *weak SDNE* as well as *expected utility NE*. The proof of the following proposition is constructive. We defer it to Appendix B.

Proposition 2. Suppose that $|N| \geq 3$. A solution φ is

(i) *implementable in strong SDNE* if and only if it is *invariant to lower SD monotonic transformations.*

(ii) *implementable in weak SDNE* if and only if it is *invariant to upper SD monotonic transformations.*

(iii) *implementable in expected utility NE* if and only if it is *invariant to Maskin monotonic transformations.*

Given the different behavioral assumptions made in defining the three notions of equilibrium and the (proper) inclusions between the sets of equilibria, it is not obvious what the logical relations between statements (i), (ii), and (iii) of Proposition 2 are. Our main result is that the three kinds of implementability are, in fact, equivalent. We introduce a fourth invariance condition, to which we will show that the above three conditions are equivalent.

For each $P_0, P_0' \in \mathcal{P}$ and each $\pi \in \Delta(A)$, we say that $\boldsymbol{P_0'}$ **is a support monotonic transformation of $\boldsymbol{P_0}$ at $\boldsymbol{\pi}$** if, for each $a \in A$ such that $\pi(a) > 0$ and each $b \in A$ such that $a \; P_0 \; b$, $a \; P_0' \; b$. As before, for each $P, P' \in \mathcal{P}^N$ and each $\pi \in \Delta(A)$, P' is a *support monotonic transformation of P at π* if for each $i \in N$, P_i' is a *support monotonic transformation of P_i at π*. The reader may find it helpful to think of P_0' as a "Maskin monotonic transformation" (Maskin 1999) of P_0 at each alternative in the support of π. Let φ be a solution.

Invariance to support monotonic transformations: For each $P \in \mathcal{P}^N$, each $\pi \in \varphi(P)$, and each $P' \in \mathcal{P}^N$, if P' is a *support monotonic transformation of* P *at* π, then $\pi \in \varphi(P')$.

Proposition 3. Let φ be a solution. The following statements about φ are equivalent.

(i) It is *invariant to lower SD monotonic transformations.*

(ii) It is *invariant to upper SD monotonic transformations.*

(iii) It is *invariant to Maskin monotonic transformations.*

(iv) It is *invariant to support monotonic transformations.*

Proof. We show that each of the first three statements is equivalent to the last.

(i)⇔(iv): We prove this by showing that for each pair $P_0, P_0' \in \mathcal{P}$ and each $\pi \in \Delta(A)$, P_0' is a *lower SD monotonic transformation* of P_0 at π if and only if it is a *support monotonic transformation* of P_0 at π. That is, $L^{sd}(P_0', \pi) \supseteq L^{sd}(P_0, \pi)$ if and only if for each pair $a, b \in A$, if $a \; P_0 \; b$ and $b \; P_0' \; a$, then $\pi(a) = 0$.

(⇒) Suppose that there is a pair $a, b \in A$ such that $a \; P_0 \; b$ and $b \; P_0' \; a$, but $\pi(a) > 0$. Let $\pi' \in \Delta(A)$ be defined by setting $\pi'(b) = \pi(b) + \pi(a), \pi'(a) = 0$, and for each $c \in A \setminus \{a, b\}$, $\pi'(c) = \pi(c)$. Then, $\pi \; P_0^{sd} \; \pi'$ and $\pi' \; P_0'^{sd} \; \pi$. That is, $\pi' \in L^{sd}(P_0, \pi) \setminus L^{sd}(P_0', \pi)$.

(⇐) Suppose that for each pair $a, b \in A$, if $a \; P_0 \; b$ and $b \; P_0' \; a$, then $\pi(a) = 0$. Let $a_1, \ldots, a_k$ be a labeling of $\text{supp}(\pi)$ such that $a_1 \; P_0 \; \ldots \; P_0 \; a_k$.[9] Then, for each $l = 1, \ldots, k$, $L(P_0', a_l) \supseteq L(P_0, a_l)$. Suppose $\pi' \in L^{sd}(P_0, \pi)$. Then,

$$\begin{array}{rcccl} \pi'(L(P_0', a_k)) & \geq & \pi'(L(P_0, a_k)) & \geq & \pi(a_k), \\ \pi'(L(P_0', a_{k-1})) & \geq & \pi'(L(P_0, a_{k-1})) & \geq & \pi(a_k) + \pi(a_{k-1}), \\ & & \vdots & & \\ \pi'(L(P_0', a_1)) & = & \pi'(L(P_0, a_1)) & = & \pi(a_k) + \pi(a_{k-1}) + \cdots + \pi(a_1). \end{array}$$

Thus, $\pi' \in L^{sd}(P_0', \pi)$.

(ii)⇔(iv): We prove this by showing that for each pair $P_0, P_0' \in \mathcal{P}$ and each $\pi \in \Delta(A)$, P_0' is an *upper SD monotonic transformation* of P_0 at π if and only if it is a *support monotonic transformation* of P_0 at π. That is, $U^{sd}(P_0', \pi) \subseteq U^{sd}(P_0, \pi)$ if and only if, for each pair $a, b \in A$, $a \; P_0 \; b$ and $b \; P_0' \; a$, implies $\pi(a) = 0$.

(⇒) Suppose that there is a pair $a, b \in A$ such that $a \; P_0 \; b$ and $b \; P_0' \; a$ but $\pi(a) > 0$. Let $\pi' \in \Delta(A)$ be defined by setting, $\pi'(b) = \pi(b) + \pi(a)$, $\pi'(a) = 0$, and for each $c \in A \setminus \{a, b\}$, $\pi'(c) = \pi(c)$. Then, $\pi \; P_0^{sd} \; \pi'$ and $\pi' \; P_0'^{sd} \; \pi$. That is, $\pi' \in U^{sd}(P_0', \pi) \setminus U^{sd}(P_0, \pi)$.

(⇐) Suppose that for each pair $a, b \in A$, if $a \; P_0 \; b$ and $b \; P_0' \; a$, then $\pi(a) = 0$. Let $a_1, \ldots, a_k$ be a labeling of $\text{supp}(\pi)$ such that $a_1 \; P_0 \; \ldots \; P_0 \; a_k$. Then, for each $l = 1, \ldots, k$, $U(P_0', a_l) \subseteq U(P_0, a_l)$. Suppose $\pi' \in U^{sd}(P_0', \pi)$. Then,

$$\begin{array}{rcl} \pi'(U(P_0, a_1)) \geq \pi'(U(P_0', a_1)) & \geq & \pi(a_1), \\ \pi'(U(P_0, a_2)) \geq \pi'(U(P_0', a_2)) & \geq & \pi(a_1) + \pi(a_2), \\ & \vdots & \\ \pi'(U(P_0, a_k)) \geq \pi'(U(P_0', a_k)) & \geq & \pi(a_1) + \pi(a_2) + \cdots + \pi(a_k). \end{array}$$

[9]For each $\pi \in \Delta(A)$, $\text{supp}(\pi) = \{a \in A : \pi(a) > 0\}$.

Thus, $\pi' \in U^{sd}(P_0, \pi)$.

(iii)⇔(iv): (⇒) We first show that for each $P_0, P_0' \in \mathcal{P}$ and each $\pi \in \Delta(A)$, if P_0' is a *support monotonic transformation* of P_0 at π, then there are $u_0 \in \mathcal{U}(P_0)$ and $u_0' \in \mathcal{U}(P_0')$ such that u_0' is a *Maskin monotonic transformation* of u_0 at π. Let $P_0, P_0' \in \mathcal{P}$ and $\pi \in \Delta(A)$. Let $u_0 \in \mathcal{U}(P_0)$. Since P_0' is a *support monotonic transformation* of P_0 at π, we can find $u_0' \in \mathcal{U}(P_0')$ such that for each $a \in \text{supp}(\pi)$, $u_0'(a) = u_0(a)$, and for each $a \notin \text{supp}(\pi)$, $u_0'(a) \leq u_0(a)$. Then, for each $\bar{\pi} \in \Delta(A)$ such that $U_0(\bar{\pi}) \leq U_0(\pi)$, $U_0'(\bar{\pi}) \leq U_0(\bar{\pi}) \leq U_0(\pi) = U_0'(\pi)$, where the first inequality and the last equality come from the construction of u_0'. Thus, u_0' is a *Maskin monotonic transformation* of u_0 at π.

Suppose that φ is *invariant to Maskin monotonic transformations*. We will show that it is *invariant to support monotonic transformations*. Let $P \in \mathcal{P}^N$ and $\pi \in \varphi(P)$. Let $P' \in \mathcal{P}^N$ be a *support monotonic transformation of P at π*. Then, there are $u \in \mathcal{U}(P)$ and $u' \in \mathcal{U}(P')$ such that u' is a *Maskin monotonic transformation* of u at π. By *invariance to Maskin monotonic transformations*, $\pi \in \varphi \circ \rho(u') = \varphi(P')$. Thus, φ is *invariant to support monotonic transformations.*

(⇐) First, we show that for each $u_0, u_0' \in \mathcal{U}$ and each $\pi \in \Delta(A)$, if u_0' is a *Maskin monotonic transformation* of u_0 at π, then $\rho(u_0')$ is a *support monotonic transformation* of $\rho(u_0)$ at π.

Let $u_0, u_0' \in \mathcal{U}$ and $\pi \in \Delta(A)$. Suppose, by contradiction, that u_0' is a *Maskin monotonic transformation* of u_0 at π, but $P_0' \equiv \rho(u_0')$ is not a *support monotonic transformation* of $P_0 \equiv \rho(u_0)$ at π. Then, there is a pair $a, b \in A$ such that $\pi(a) > 0$ and $a \ P_0 \ b$, but $b \ P_0' \ a$. Let $\pi' \in \Delta(A)$ be such that $\pi'(a) = 0$, $\pi'(b) = \pi(a) + \pi(b)$, and for each $c \in A \setminus \{a, b\}$, $\pi'(c) = \pi(c)$. It is easily checked that $U_0(\pi') < U_0(\pi)$ and $U_0'(\pi') > U_0'(\pi)$, a contradiction.

Suppose that φ is *invariant to support monotonic transformations.* Let $u \in \mathcal{U}^N$ and $\pi \in \varphi \circ \rho(u)$. Let $u' \in \mathcal{U}^N$ be a *Maskin monotonic transformation of u at π*. Then, $\rho(u')$ is a *Maskin monotonic transformation* of $\rho(u')$ at π. By *invariance to support monotonic transformations*, $\pi \in \varphi \circ \rho(u')$. □

Recall that we are "extending" ordinal preferences over sure alternatives to lotteries using the SD relation. Intuitively, the equivalence of the four seemingly different invariance conditions is due to the "sparseness" of the domain of preferences. Let $A \equiv \{a, b, c\}$, $\pi \in \Delta(A)$, and $P_0 \in \mathcal{P}$ where $a \ P_0 \ b \ P_0 \ c$. For $P_0' \in \mathcal{P} \setminus \{P_0\}$, to be a *lower SD* or *upper SD monotonic transformation* of P_0 at π, π needs to be on the boundary of the simplex (see Figures 1(a) and 1(b)). Similarly, if $u_0 \in \mathcal{U}$ is such that $u_0(a) > u_0(b) > u_0(c)$, and $u_0' \in \mathcal{U} \setminus \{u_0\}$, is a *Maskin monotonic transformation* of u_0 at π, then π needs to be on the boundary of the simplex (see Figure 1(c)).[10]

Our main theorem is a consequence of Propositions 2 and 3.

Theorem 1. Suppose that $|N| \geq 3$. Let φ be a solution. The following statements about φ are equivalent.

(i) It is *implementable in strong SDNE.*

(ii) It is *implementable in weak SDNE.*

(iii) It is *implementable in expected utility NE.*

[10] That there are no monotonic transformations of a preference relation on the interior of the simplex renders *Maskin-invariance* vacuous on all but the boundary. This is a key insight for the "anything goes" results on *virtual implementation* (Matsushima 1988, Abreu and Sen 1991).

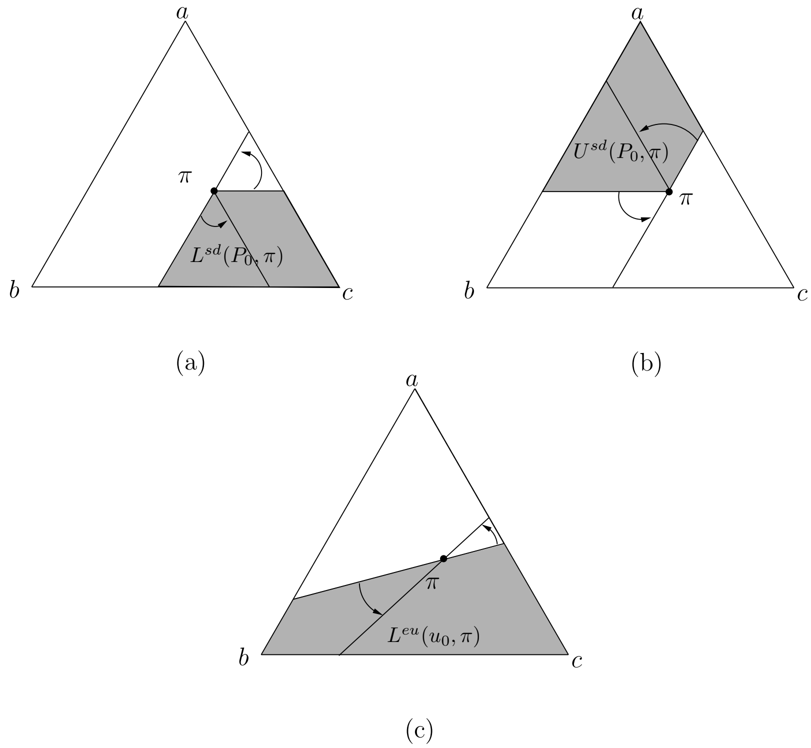

Figure 1: **Monotonic transformations only exist on the boundary of the simplex.** Consider $P_0 \in \mathcal{P}$ such that $a\ P_0\ b\ P_0\ c$ and an interior $\pi \in \Delta(A)$. (a) The *SD lower contour set* of P_0 at π is shaded. For any $P'_0 \in \mathcal{P} \setminus \{P_0\}$ (say, $b\ P'_0\ a\ P'_0\ c$), we obtain the *SD lower contour set* of P'_0 at π by appropriately rotating the "V" shaped boundary so that its sides are parallel to two of the faces of the simplex. (b) The *SD upper contour set* of P_0 at π is shaded. Again, for any $P'_0 \in \mathcal{P} \setminus \{P_0\}$ (say, $b\ P'_0\ a\ P'_0\ c$), we obtain the *SD upper contour set* of P'_0 at π by rotating the "V" shaped boundary. (c) For $u_0 \in \mathcal{U}$ such that $u_0(a) > u_0(b) > u_0(c)$, the *expected utility lower contour set* of u_0 at π is shaded. For any $u'_0 \in \mathcal{U} \setminus \{u_0\}$, we obtain the *expected utility lower contour set* of u'_0 at π by rotating a line through π. For each of (a), (b), and (c), the inclusion relation between these sets is impossible when π is on the interior of $\Delta(A)$.

Since *invariance to support monotonic transformation* is necessary and sufficient for implementability in each kind of Nash equilibrium, a natural question is whether it is preserved under intersection, union, or convex combination of solutions. The reader familiar with the implementation literature will know that such invariance conditions are commonly preserved under intersection and union. That it is preserved under convex combination is different from usual results.

Proposition 4.

(i) Let Φ be a collection of solutions that are *invariant to support monotonic transformations*. Then, $\bigcap_{\phi \in \Phi} \phi$ (when it is nonempty-valued) and $\bigcup_{\phi \in \Phi} \phi$ are *invariant to support monotonic transformations.*

(ii) If ϕ and φ are *invariant to support monotonic transformations*, then for each $\alpha \in (0, 1)$, $(\alpha\phi + (1 - \alpha)\varphi)$ is *invariant to support monotonic transformations*

We defer the proof of Proposition 4 to Appendix C.

Remark 3. Since *invariance to support monotonic transformations* is preserved under arbitrary intersection, it is possible to compute the "minimal invariant enlargement" of a solution that violates it (Sen 1995, Thomson 1999).

We close this section with an example to show that *invariance to support monotonic transformations* is not a vacuous property. For each $P_0 \in \mathcal{P}$ and each $\pi \in \Delta(A)$, let $C(P_0, \pi)$ be the upper contour set of P_0 at the worst alternative in supp(π). That is, $C(P_0, \pi) \equiv \bigcup_{a \in \text{supp}(\pi)} U(P_0, a)$. The solution in the following example selects every lottery that minimizes the size of the largest such set among all agents.

Example 1. For each $P \in \mathcal{P}^N$, let $\varphi(P) \equiv \text{argmin}_{\pi \in \Delta(A)}\{\max_{i \in N}|C(P_i, \pi)|\}$. Let $N = \{1, 2, 3\}$, $A \equiv \{a, b, c\}$, and let $P_1, P_1', P_2, P_3 \in \mathcal{P}$ be such that:

P_1	P_1'	P_2	P_3
a	a	b	a
b	c	c	c
c	b	a	b

Let $\pi \in \Delta(A)$ be such that $\pi(a) = \frac{1}{2}, \pi(b) = 0$, and $\pi(c) = \frac{1}{2}$. It is easy to check that $\pi \in \varphi(P_1, P_2, P_3)$. Further, $\varphi(P_1', P_2, P_3) = \{\pi'\}$ where $\pi'(c) = 1$. However, P_1' is a *support monotonic transformation* of P_1 at π, yet $\pi \notin \varphi(P_1', P_2, P_3)$.

5 Economic environments

In Section 2 we assumed that preferences over sure social alternatives are strict. This precludes economic environments where each agent cares only about his component of an allocation. In this section, we define a broad class of economic environments to which our results generalize. The abstract setting of Section 2 is a special case.

Let O be a finite set of objects and N be a set of agents such that $|N| \geq 3$. For each $o \in O$, let $q_o \in \mathbb{Z}_+$ be the **availability** of o.

Each $i \in N$ consumes a bundle $x_i \in \mathbb{Z}_+^O$. The set of admissible bundles for i is restricted to his **consumption set**, $\boldsymbol{X_i} \subseteq \mathbb{Z}_+^O$, and he ranks the alternatives in X_i according to a linear order P_i.[11] We assume that $|X_i| \geq 3$. For each pair $x_i, y_i \in X_i$, as before, we write $x_i \; P_i \; y_i$ if i prefers x_i to y_i. Let $\mathcal{P}_i$ be the set of such linear orders over X_i. Thus $\times_{i \in N}\mathcal{P}_i$ is the set of all preference profiles. As in Section 2, for each $P_i \in \mathcal{P}_i$ and each $x_i \in X_i$, let $U(P_i, x_i)$ be the upper contour set of P_i at x_i and let $L(P_i, x_i)$ be the lower contour set.

Remark 4. In many economic environments like classical exchange, monotonicity of preferences renders the "no veto" condition vacuous. Thus, for non-probabilistic models or probabilistic models with complete preferences, that Maskin's invariance is both necessary and sufficient for implementability in NE follows from Maskin (1999). However, we do not assume that preferences are monotonic. Thus, the *no veto* condition is not vacuous.

Just as in Section 2, let $\mathcal{U}_i$ be the set of all Bernoulli utility functions without indifference defined over X_i. For each $P_i \in \mathcal{P}_i$, let $\mathcal{U}(P_i)$ be the Bernoulli utility functions consistent with P_i. For each $P \in \times_{i \in N}\mathcal{P}_i$, let $\mathcal{U}(P) \equiv \mathcal{U}(P_1) \times \cdots \times \mathcal{U}(P_n)$.

[11]Note that X_i is the set of *deterministic* allocations that i may receive.

A lottery for i is $\pi_i \in \Delta(X_i)$. The SD extension of $P_i \in \mathcal{P}_i$ to $\Delta(X_i)$ is just as in Section 2. For each $\pi_i \in \Delta(X_i)$, we similarly define the SD upper and lower contour sets, $U^{sd}(P_i, \pi_i)$ and $L^{sd}(P_i, \pi_i)$ as well as, for each $u_0 \in \mathcal{U}_i$, the expected utility lower contour set $L^{eu}(u_i, \pi_i)$.

A feasible allocation specifies an admissible bundle for each agent subject to the availability of each object. The set of feasible allocations is

$$\mathbf{F} \equiv \{x \in \times_{i \in N} X_i : \text{ for each } o \in O, \sum_{i \in N} x_{io} \leq q_o\}.$$

We assume that for each $i \in N$, the projection of F to the i^{th} coordinate is X_i. That is, i can consume every bundle in his consumption set at some feasible allocation.[12]

This model covers the probabilistic assignment of single objects (Hylland and Zeckhauser 1979, Bogomolnaia and Moulin 2001), multiple objects (Kojima 2009), and packages (Budish 2011, Budish et al. 2013, Peivandi 2014).

The set of **probabilistic allocations** is $\Delta(F)$. For each $\pi \in \Delta(F)$ and each $i \in N$, π_i is the marginal distribution of the i^{th} component. That is, $\pi_i \in \Delta(X_i)$ is such that for each $x_i \in X_i$, $\pi_i(x_i) = \pi(\{y \in F : y_i = x_i\})$.

For each pair $\pi, \pi' \in \Delta(F)$, each $i \in N$ compares only the i^{th} component of each: π_i and π'_i. Thus, if π_i and π'_i are identical, then i is indifferent between π and π'.

A solution is a nonempty-valued correspondence $\varphi : \times_{i \in N} \mathcal{P}_i \rightrightarrows \Delta(F)$. We restrict attention to **regular** solutions: For each $P \in \times_{i \in N} \mathcal{P}_i$, and each $\pi \in \varphi(P)$, given $i \in N$, if for each $j \in N \setminus \{i\}$, $|\text{supp}(\pi_j)| = 1$, then $|\text{supp}(\pi_i)| = 1$.[13]

Replacing A with F, the definitions of a game form, a game, and the three notions of equilibrium are analogous to those in Section 2. The definitions of the invariance conditions of Section 3 need no modification. However, in Section 2, preferences over A are assumed to be strict. This is why Proposition 2 does not apply directly. Nonetheless, it holds with small adjustments in the proof. With Proposition 2 in hand, Theorem 1 holds even in economic environments.

6 Application to probabilistic assignment

As a concrete example of an economic environment, we consider the problem of probabilistic assignment of single objects (Hylland and Zeckhauser 1979, Bogomolnaia and Moulin 2001): For each $o \in O$, $q_o = 1$, $|O| = |N|$, and for each $i \in N$,

$$X_i = \{x_i \in \mathbb{Z}^O_+ : \text{ for each } o \in O, x_{io} \in \{0, 1\} \text{ and } \sum_{o \in O} x_{io} = 1\}.$$

That is, the objects are unique and each agent is assigned one of them. As it will cause no confusion, we will refer to $x_i \in X_i$, where $x_{io} = 1$, as o. For this model *regularity* is vacuous. For each $i \in N$, an admissible preference is a linear order over O. We denote the set of linear orders by $\mathcal{P}$.

[12] To see that the environment of Section 2 is a special case, note that we can set, for each $i \in N, X_i \equiv A$ and let $F \equiv \{x \in A^N : \text{ for each pair } i, j \in N, x_i = x_j\}$.

[13] *Regularity* is a mild requirement. It is vacuous if every agent can be assigned only one object and there are exactly as many objects as there are agents. In general, it is implied by very weak efficiency requirements: Suppose that every agent but i receives some objects with probability one. Since the remaining objects are available with certainty, what i receives should be certain as well.

In this section, our goal is to show that the invariance conditions of Section 3, and therefore the implementability requirements, are mild and compatible with many other normative properties of a solution.

Since the three implementability requirements are equivalent, we say that the solution is **implementable in NE** if a solution is implementable in any one of the three kinds of equilibrium.

We first define the SD Pareto solution. We say that an allocation is *SD Pareto efficient* if no other allocation makes at least one agent better off in the SD sense while each other agent is either made better off or his component of the allocation is unchanged (Bogomolnaia and Moulin 2001).[14]

The **SD Pareto solution, $\boldsymbol{PE^{sd}}$:** For each $P \in \mathcal{P}^N, PE^{sd}(P) \equiv \{\pi \in \Delta(F)$: there is no $\pi' \in \Delta(F) \setminus \{\pi\}$ such that for each $i \in N$, $\pi'_i \; R_i^{sd} \; \pi_i\}$. That is, $PE^{sd}(P)$ is the set of *SD Pareto efficient* allocations at P.

The next solution selects every allocation with the property that not only can each agent compare his allocation to anybody else's in the SD sense, but he finds it to be at least as desirable (Foley 1967, Bogomolnaia and Moulin 2001).

The **SD no-envy solution, $\boldsymbol{F^{sd}}$**: For each $P \in \mathcal{P}^N$, $F^{sd}(P) \equiv \{\pi \in \Delta(F)$: for each $i, j \in N$, $\pi_i \; R_i^{sd} \; \pi_j\}$.

The following solution is parameterized by a private endowment or reference assignment $\omega \in \Delta(F)$. For each $i \in N$, ω_i is a lower bound on i's welfare: he ought to be assigned a lottery that he finds at least as desirable as ω_i in the SD sense (Athanassoglou and Sethuraman 2011).[15]

The **SD $\boldsymbol{\omega}$-lower bound solution, $\underline{\boldsymbol{B}}_{\boldsymbol{\omega}}^{\boldsymbol{sd}}$**: For each $P \in \mathcal{P}^N$, $\underline{B}_\omega^{sd}(P) \equiv \{\pi \in \Delta(F)$: for each $i \in N$, $\pi_i \; R_i^{sd} \; \omega_i\}$.

Next is a single valued solution that holds a central place in the literature as a selection from $PE^{sd} \cap F^{sd}$.[16]

Bogomolnaia-Moulin solution, $\boldsymbol{BM}$: For each $P \in \mathcal{P}^N$, $BM(P)$ is the output of the *Bogomolnaia-Moulin* algorithm with input P.

The following proposition shows that the solutions that we have discussed above are implementable in NE. Moreover, for three-agent economies, BM is the only selection from $PE^{sd} \cap F^{sd}$ that is implementable in NE.[17]

Proposition 5.

(i) The solutions PE^{sd}, F^{sd}, for each $\omega \in \Delta(F)$, $\underline{B}_\omega^{sd}$, and BM are *implementable in NE.*

(ii) For $|N| = 3$, BM is the only selection from $PE^{sd} \cap F^{sd}$ that is *implementable in NE*

Proof. (i) We show that each of these solutions is *invariant to support monotonic transformation.* $\boldsymbol{PE^{sd}}$: Let $P \in \mathcal{P}^N$, $\pi \in PE^{sd}(P)$, and $P' \in \mathcal{P}^N$. Suppose for the sake of contradiction, that P' is a *support monotonic transformation of* P *at* π and that $\pi \notin PE^{sd}(P')$. Then there are a list of agents $(i_1, i_2, \cdots, i_k)$ and a list of objects $(o_1, o_2, \cdots, o_k)$ such that for each $l \in \{1, \cdots, k\}$,

[14] This is often called *ordinal efficiency.* We have adopted the terminology and the notation of Thomson (2013).
[15] If for each $i \in N$ and each $a \in A$, $\omega_i(a) = \frac{1}{n}$, this is the "SD equal division lower bound" solution (Heo 2014).
[16] See Bogomolnaia and Moulin (2001) for a formal definition of this solution.
[17] Proposition 5 (ii) does not generalize to more than three agents.

$\pi_{i_l}(o_{(l+1) \bmod k}) > 0$ and $o_l \; P'_{i_l} \; o_{(l+1) \bmod k}$ (Bogomolnaia and Moulin 2001). Since $P' \in \mathcal{P}^N$ is a *support monotonic transformation of* P *at* π, for each $l \in \{1, \cdots, k\}$, $o_l \; P_{i_l} \; o_{(l+1) \bmod k}$. This contradicts $\pi \in PE^{sd}(P)$.

$\boldsymbol{F^{sd}}$: Let $P \in \mathcal{P}^N$ and $\pi \in F^{sd}(P)$. Then, for each pair $i, j \in N$, $\pi_i \; R_i^{sd} \; \pi_j$. That is, for each $o \in O$ such that $\pi_i(o) > 0$, $\pi_i(L(P_i, o)) \leq \pi_j(L(P_i, o))$. Let $P' \in \mathcal{P}^N$ be a *support monotonic transformation of* P *at* π. Then, for each $k \in N$ and each $o \in O$ such that $\pi_k(o) > 0$, $L(P_k, o) \subseteq L(P'_k, o)$. Thus, for each pair $i, j \in N$ and each $o \in O$ such that $\pi_i(o) > 0$, $\pi_j(L(P'_i, o)) \geq \pi_j(L(P_i, o)) \geq \pi_i(L(P_i, o)) = \pi_i(L(P'_i, o))$. The last equality comes from the fact that P'_i is a *support monotonic transformation* of P_i at π. Thus, $\pi_i \; R_i'^{sd} \; \pi_j$ and so $\pi \in F^{sd}(P')$.

$\boldsymbol{\underline{B}^{sd}_{\omega}}$: Let $P \in \mathcal{P}^N$ and $\pi \in \underline{B}^{sd}_{\omega}(P)$. Then, for each $i \in N$, $\pi_i \; R_i^{sd} \; \omega_i$. Let $P' \in \mathcal{P}^N$ be a *support monotonic transformation of* P *at* π. Then, for each $i \in N$ and each $o \in O$ with $\pi_i(o) > 0$, $L(P_i, o) \subseteq L(P'_i, o)$. Thus, for each $i \in N$ and each $o \in O$ such that $\pi_i(o) > 0$, $\omega_i(L(P'_i, o)) \geq \omega_i(L(P_i, o)) \geq \pi_i(L(P_i, o)) = \pi_i(L(P'_i, o))$. Thus, $\pi_i \; R_i'^{sd} \; \omega_i$ and so $\pi \in \underline{B}^{sd}_{\omega}(P')$.

BM: It is straightforward to see that the BM solution is *invariant to support monotonic transformations*, so we omit the proof.

(ii) Since BM is a selection from $PE^{sd} \cap F^{sd}$ (Bogomolnaia and Moulin 2001) and is *invariant to support monotonic transformations*, it suffices to show that if φ is a selection from $PE^{sd} \cap F^{sd}$ that is *invariant to support monotonic transformations*, then $\varphi \equiv BM$. Let $N \equiv \{1, 2, 3\}$, $O \equiv \{a, b, c\}$. Up to relabeling of agents and objects, the following list of (partial) profiles of preferences is exhaustive:

Profile 1:			Profile 2:			Profile 3:			Profile 4:			Profile 5:			Profile 6:		
P_1	P_2	P_3	P_1	P_2	P_3	P_1	P_2	P_3	P_1	P_2	P_3	P_1	P_2	P_3	P_1	P_2	P_3
a	b	c	a	a	a	a	a	a	a	a	b	a	a	b	a	a	b
·	·	·	b	b	b	b	b	c	c	c	·	b	b	·	b	c	·
·	·	·	c	c	c	c	c	b	b	b	·	c	c	·	c	b	·

Profiles 1-4: Since φ is a selection from $PE^{sd} \cap F^{sd}$, the choice of φ is pinned down and coincides with the recommendation made by the BM solution.[18]

Profile 5: Let $P_3 \in \mathcal{P}$ be such that $b \; P_3 \; a \; P_3 \; c$. Let $\pi = \varphi(P)$. Since $\pi \in F^{sd}(P)$, $\pi_1(c) = \pi_2(c) = \pi_3(c) = \frac{1}{3}$ and $\pi_1(a) = \pi_2(a)$. Since $\pi \in PE^{sd}(P)$, $\pi_3(a) = 0$. Thus, $\pi_1 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3}), \pi_2 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$, and $\pi_3 = (0, \frac{2}{3}, \frac{1}{3})$. Thus, $\varphi(P) = BM(P)$.

Now, let $P'_3 \in \mathcal{P}$ be such that $b \; P'_3 \; c \; P'_3 \; a$. By *invariance to support monotonic transformations*, since $\pi_3(a) = 0$ we have that $\varphi(P_1, P_2, P'_3) = \varphi(P) = \pi$. By definition of BM, $BM(P_1, P_2, P'_3) = \pi$. Thus $\varphi(P_1, P_2, P'_3) = \pi = BM(P_1, P_2, P'_3)$.

Profile 6: Let $P_3 \in \mathcal{P}$ be such that $b \; P_3 \; a \; P_3 \; c$. Let $\pi \equiv \varphi(P)$. Since $\pi \in F^{sd}(P)$, $\pi_1(a) = \pi_2(a)$ and $\pi_1(a) + \pi_1(b) = \pi_3(a) + \pi_3(b)$. Since $\pi \in PE^{sd}(P)$, $\pi_3(a) = \pi_2(b) = 0$. Thus, $\pi_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $\pi_2 = (\frac{1}{2}, 0, \frac{1}{2})$, and $\pi_3 = (0, \frac{3}{4}, \frac{1}{4})$. Thus, $\varphi(P) = BM(P)$.

Now, let $P'_3 \in \mathcal{P}$ be such that $b \; P'_3 \; c \; P'_3 \; a$. By *invariance to support monotonic transformations*, since $\pi_3(a) = 0$ we have that $\varphi(P_1, P_2, P'_3) = \varphi(P) = \pi$. By definition of BM, $BM(P_1, P_2, P'_3) = \pi$. Thus $\varphi(P_1, P_2, P'_3) = \pi = BM(P_1, P_2, P'_3)$. □

As a corollary to Propositions 4 and 5 we see that more solutions of interest are imple-

[18] See the proof of Proposition 2 in Bogomolnaia and Moulin (2001).

mentable.[19] Take, for instance, $PE^{sd} \cap F^{sd}$ and for each $\omega \in \Delta(F)$, $PE^{sd} \cap \underline{B}^{sd}_{\omega}$, both of which are nonempty-valued (Bogomolnaia and Moulin 2001, Athanassoglou and Sethuraman 2011). Earlier results have shown that for every selection from $PE^{sd} \cap F^{sd}$, there are a profile of preferences and an agent who gains in the SD sense by misreporting his preferences at that profile (Bogomolnaia and Moulin 2001).[20] The same problem occurs for every selection from $PE^{sd} \cap \underline{B}^{sd}_{\omega}$ (Athanassoglou and Sethuraman 2011). Our results say that for each of the above mentioned solutions we can construct a game in which many agents (perhaps all) may misreport their preferences and yet we achieve as NE outcomes those and only those lotteries recommended by that solution.

We close this section with an example to show that *invariance to support monotonic transformations* is not a vacuous property even in this restricted environment. As in Example 1, we define for each $i \in N$, each $P_i \in \mathcal{P}$ and each $\pi \in \Delta(F)$, $C(P_i, \pi_i) \equiv \bigcup_{o \in \text{supp}(\pi_i)} U(P_i, o)$.

Example 2. For each $P \in \mathcal{P}^N$, let $\varphi(P) \equiv \text{argmin}_{\pi \in \Delta(F)}\{\max_{i \in N}|C(P_i, \pi_i)|\}$. Let $N = \{1, 2, 3\}$, $O \equiv \{a, b, c\}$, and let the pair $P, P' \in \mathcal{P}^N$ be such that:

P_1	P_2	P_3
a	a	b
b	b	a
c	c	c

P'_1	P'_2	P'_3
a	a	b
b	b	c
c	c	a

Let $\pi \in \Delta(F)$ be such that $\pi_1 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3}), \pi_2 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$, and $\pi_3 = (0, \frac{2}{3}, \frac{1}{3})$. It is easy to check that $\pi \in \varphi(P)$. Further, for each $\pi' \in \varphi(P')$, $\pi'_3(c) = 1$. However, for each $i \in N$, P'_i is a *support monotonic transformation* of P_i at π_i, yet $\pi \notin \varphi(P')$.

Appendices

A The inclusion relations of Proposition 1 may be strict

For each $P \in \mathcal{P}^N$ and each $a \in A$, let $N(a, P)$ be the set of agents who most prefer a. That is, $N(a, P) \equiv \{i \in N : a \; P_i \; b \text{ for each } b \in A \setminus \{a\}\}$. Let $M(p)$ be the set plurality-winners: Each is most preferred by at least as many agents as any other alternative. That is, $M(P) \equiv \{a \in A : |N(a, P)| \geq |N(b, P)| \text{ for each } b \in A\}$.

Let φ be the single valued solution that picks each plurality winner with the same probability. That is, for each $P \in \mathcal{P}^N$,

$$\varphi(P)(a) \equiv \begin{cases} \frac{1}{|M(P)|}, & \text{if } a \in M(P), \\ 0, & \text{otherwise.} \end{cases}$$

Let $N \equiv \{1, 2, 3\}$, $A \equiv \{a, b, c\}$, $P \in \mathcal{P}^N$, and $u \in \mathcal{U}(P)$ be such that

P_1	P_2	P_3
a	b	c
b	c	a
c	a	b

$$\begin{array}{lll} u_1(a) = 100 & u_2(a) = 0 & u_3(a) = 99 \\ u_1(b) = 1 & u_2(b) = 100 & u_3(b) = 0 \\ u_1(c) = 0 & u_2(c) = 1 & u_3(c) = 100 \end{array}$$

[19]We also check implementability of some other solutions in Appendix E.

[20]When agents may receive more than one object, even stronger impossibility results hold (Kojima 2009, Kasajima 2011, Heo 2014).

Consider the preference revelation game induced by φ, $\Gamma \equiv (\mathcal{P}^N, \varphi)$. It is easy to check that for each $i \in N$, $(P_i, P_i, P_i) \in SNE^{sd}(\Gamma, P)$ since the outcome does not change no matter how an agent deviates.

Next, we show that $(P_1, P_2, P_1) \in NE(\Gamma, u) \setminus SNE^{sd}(\Gamma, P)$. Note that $\varphi(P_1, P_2, P_1) = (1, 0, 0)$ and if agent 3 reports P_3, then $\varphi(P_1, P_2, P_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Therefore, $(P_1, P_2, P_1) \notin SNE^{sd}(\Gamma, P)$. We now show that $(P_1, P_2, P_1) \in NE(\Gamma, u)$. First, the outcome does not change no matter how agent 2 deviates. Second, agent 1 does not have an incentive to deviate from (P_1, P_2, P_1) because $\varphi(P_1, P_2, P_1) = (1, 0, 0)$ is his most preferred outcome at P_1. Third, agent 3 does not have an incentive to report b as his favorite alternative since the resulting outcome $(0, 1, 0)$ is his least preferred outcome at P_3. Agent 3 does not have an incentive to report c as his favorite alternative either since the outcome changes from $(1, 0, 0)$ to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and his expected utility decreases.

Lastly, we show that $(P_2, P_2, P_3) \in WNE^{sd}(\Gamma, P) \setminus NE(\Gamma, u)$. Note that $\varphi(P_2, P_2, P_3) = (0, 1, 0)$ and if agent 1 reports P_1, then $\varphi(P_1, P_2, P_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. It is easy to check that agent 1's expected utility increases by doing so. Therefore, $(P_2, P_2, P_3) \notin NE(\Gamma, u)$. We now show that $(P_2, P_2, P_3) \in WNE(\Gamma, P)$. First, the outcome does not change no matter how agent 3 deviates. Second, agent 2 does not have an incentive to deviate because $\varphi(P_2, P_2, P_3) = (0, 1, 0)$ is his most preferred outcome at P_2. Third, agent 1 does not have an incentive to report c as his favorite alternative since the resulting outcome $(0, 0, 1)$ is his least preferred outcome at P_1. If agent 1 reports a as his favorite alternative, then the resulting outcome is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ which does not stochastically dominate $(0, 1, 0)$ at P_1. Therefore, $(P_2, P_2, P_3) \in WNE(\Gamma, P)$.

B Proof of Proposition 2

We prove part (i) of Proposition 2: a solution φ is *implementable in strong SDNE* if and only if it is *invariant to lower SD monotonic transformations*. The proofs of parts (ii) and (iii) are very similar. We omit them but highlight the differences at the end.

(⇒) Suppose that φ is *implementable in strong SDNE*. Suppose that it is implemented by $\Gamma \equiv (S, h)$. Let $P \in \mathcal{P}^N$ and $\pi \in \varphi(P)$. Then, there is $s \in SNE^{sd}(\Gamma, P)$ such that $h(s) = \pi$. Since $s \in SNE^{sd}(\Gamma, P)$, for each $i \in N$ and each $s_i' \in S_i$, $h(s_i', s_{-i}) \in L^{sd}(P_i, \pi)$.

Let $P' \in \mathcal{P}^N$ be such that for each $i \in N$, $L^{sd}(P_i', \pi) \supseteq L^{sd}(P_i, \pi)$. Then, for each $s_i' \in S_i$, $h(s_i', s_{-i}) \in L^{sd}(P_i', \pi)$. Thus, $s \in SNE^{sd}(\Gamma, P')$ so $\pi = h(s) \in \varphi(P')$.

(⇐) In this part of the proof we define a game form and show that it *implements* φ *in strong SDNE*.

First, we present some notation. For each $P_0 \in \mathcal{P}$ and each $B \subseteq A$, let $\boldsymbol{\tau(P_0, B)}$ be the **top alternative under $\boldsymbol{P_0}$ in $\boldsymbol{B}$**. That is, $\tau(P_0, B) = a$ if and only if $a \in B$ and for each $b \in B \setminus \{a\}$, $a\ P_0\ b$. For convenience, we usually drop A in the expression $\tau(P_0, A)$ and write $\tau(P_0)$ instead. That is, when the second argument is omitted, it is A. For each $a \in A$, let $\boldsymbol{\delta^a}$ $\in \Delta(A)$ be such that $\delta^a(a) = 1$. Let $\boldsymbol{D} \equiv \{\delta^a : a \in A\}$. Let $\boldsymbol{\bar{\pi}} \in \Delta(A)$ be such that for each $a \in A$, $\bar{\pi}(a) \equiv \frac{1}{|A|}$.

Next, we define a canonical game form. For each $i \in N$, let $S_i \equiv \mathcal{P}^N \times \Delta(A) \times \Delta(A) \times \mathbb{N}_+$. Denote a generic member of S_i by $s_i \equiv (P^i, \pi^i, \sigma^i, n^i)$. Define $h : S \to \Delta(A)$ by setting, for each

$s \in S$,

$$
h(s) \equiv \begin{cases}
\pi & \text{if for each } i \in N,\ s_i = (P, \pi, \sigma, n) \text{ and } \pi \in \varphi(P) \\[2ex]
\sigma^i & \left\{ \begin{array}{l} \text{if there is } i \in N \text{ such that} \\ \text{for each } j \in N \setminus \{i\}, s_j = (P, \pi, \sigma, n) \neq s_i = (P^i, \pi^i, \sigma^i, n^i) \\ \text{and } \sigma^i \in L^{sd}(P_i, \pi) \setminus D,^{21} \end{array} \right. \\[6ex]
\pi & \left\{ \begin{array}{l} \text{if there is } i \in N \text{ such that} \\ \text{for each } j \in N \setminus \{i\}, s_j = (P, \pi, \sigma, n) \neq s_i = (P^i, \pi^i, \sigma^i, n^i) \\ \text{and } \sigma^i \notin L^{sd}(P_i, \pi) \setminus D, \end{array} \right. \\[6ex]
\frac{1}{n^{i^*}+1}\sigma^{i^*} + \frac{n^{i^*}}{n^{i^*}+1}\pi^{i^*} & \left\{ \begin{array}{l} \text{if there is a triple } i, j, k \in N \text{ such that } s_i \neq s_j, s_i \neq s_k, s_j \neq s_k, \\ \text{and } \pi^{i^*} \neq \sigma^{i^*}, \text{ where } i^* \equiv \min \left\{ \arg\max_{j \in N} n^j \right\}, \text{ and} \end{array} \right. \\[6ex]
\bar{\pi} & \text{otherwise.}
\end{cases}
$$

Let $\Gamma \equiv (S, h)$.[22] Our goal is to show that for each $P \in \mathcal{P}^N$, $h(SNE^{sd}(\Gamma, P)) = \varphi(P)$.

First, we show that $\varphi(P) \subseteq h(SNE^{sd}(\Gamma, P))$. Let $\pi \in \varphi(P)$. Let $s \in S$ be such that for each $i \in N$, $s_i = (P, \pi, \pi, 0)$. For each $i \in N$ and each $s'_i \in S_i \setminus \{s_i\}$, either $h(s'_i, s_{-i}) = h(s) = \pi$ or $h(s'_i, s_{-i}) \in L^{sd}(P_i, \pi) \setminus D$. Thus, $s \in SNE^{sd}(\Gamma, P)$. Consequently $\pi = h(s) \in h(SNE^{sd}(\Gamma, P))$.

Next, consider $\overset{\mathrm{T}}{P} \in \mathcal{P}^N$ and $s \in SNE^{sd}(\Gamma, \overset{\mathrm{T}}{P})$. We complete the proof by showing that $h(s) \in \varphi(\overset{\mathrm{T}}{P})$. Let $\pi \equiv h(s)$.

Recall that for each $P_0 \in \mathcal{P}$ and each $\pi \in \Delta(A)$, $L^{sd}(P_0, \pi)$ is closed and convex. Thus, for each $P'_0 \in \mathcal{P}$ if $L^{sd}(P'_0, \pi) \supseteq L^{sd}(P_0, \pi) \setminus D$ then $L^{sd}(P'_0, \pi) \supseteq L^{sd}(P_0, \pi)$.[23]

Case 1: For each $i \in N$, $s_i = (P, \pi, \sigma, n)$ and $\pi \in \varphi(P)$. Then, for each $i \in N$, $L^{sd}(\overset{\mathrm{T}}{P}_i, \pi) \supseteq L^{sd}(P_i, \pi) \setminus D$: Otherwise, there is $\sigma^i \in \Delta(A)$ such that $\sigma^i \in L^{sd}(P_i, \pi) \setminus (D \cup L^{sd}(\overset{\mathrm{T}}{P}_i, \pi))$. Let $s'_i \equiv (P, \pi, \sigma^i, n)$. Then $h(s'_i, s_{-i}) = \sigma^i$. So $h(s'_i, s_{-i}) = \sigma^i \notin L^{sd}(\overset{\mathrm{T}}{P}_i, \pi) = L^{sd}(\overset{\mathrm{T}}{P}_i, h(s))$. This contradicts $s \in SNE^{sd}(\Gamma, \overset{\mathrm{T}}{P})$. Since $\pi \in \varphi(P)$ and for each $i \in N$, $L^{sd}(\overset{\mathrm{T}}{P}_i, \pi) \supseteq L^{sd}(P_i, \pi)$, by *invariance to lower SD monotonic transformations*, $\pi \in \varphi(\overset{\mathrm{T}}{P})$.

Case 2: There is $i \in N$ such that, for each $j \in N \setminus \{i\}, s_j = (P, \pi, \sigma, n) \neq s_i$. Let $\tilde{\pi} \equiv h(s)$. Then for each $j \in N \setminus \{i\}$, $\tilde{\pi} = \delta^{\tau(\overset{\mathrm{T}}{P}_j)}$: otherwise, let $\hat{s}_j \in S_j$ be such that $\hat{n}^j \equiv \max\{n^i, n\} + 1$, $\hat{\sigma}^j \equiv \tilde{\pi}$, $\hat{P}^j \in \mathcal{P}^N$, and $\hat{\pi}^j = \delta^{\tau(\overset{\mathrm{T}}{P}_j)}$. Then, $h(\hat{s}_j, s_{-j}) = \frac{1}{\hat{n}^j+1}\tilde{\pi} + \frac{\hat{n}^j}{\hat{n}^j+1}\delta^{\tau(\overset{\mathrm{T}}{P}_j)} \ \overset{\mathrm{T}}{P}{}^{sd}_j \ \tilde{\pi}$. This contradicts $s \in SNE^{sd}(\Gamma, \overset{\mathrm{T}}{P})$. Thus, for each $j \in N \setminus \{i\}$, $\tilde{\pi} = \delta^{\tau(\overset{\mathrm{T}}{P}_j)}$.[24]

[21] Equivalently: $|\text{supp}(\sigma^i)| \neq 1$ and $\pi \ R_i^{sd} \ \sigma^i$.

[22] This is a variant of the canonical game form by Maskin (1999).

[23] Otherwise, there is $a \in A$ such that $\delta^a \in L^{sd}(P_0, \pi)$ but $\delta^a \notin L^{sd}(P'_0, \pi)$. Therefore, $\delta^a \neq \pi$. Since $L^{sd}(P_0, \pi)$ is closed and convex and $\pi \neq \delta^a$, for each $\alpha \in (0, 1)$, $\alpha\delta^a + (1 - \alpha)\pi \in L^{sd}(P_0, \pi) \setminus D \subseteq L^{sd}(P'_0, \pi)$, where the last inclusion relation is from the assumption. Since $\delta^a \notin L^{sd}(P'_0, \pi)$ and $L^{sd}(P'_0, \pi)$ is closed, there is $\beta \in (0, 1)$ such that $\beta\delta^a + (1 - \beta)\pi \in L^{sd}(P_0, \pi) \setminus (L^{sd}(P'_0, \pi) \setminus D)$, a contradiction.

[24] That is, $\tilde{\pi}$ is the top alternative for each $j \in N \setminus \{i\}$ under $\overset{\mathrm{T}}{P}_j$.

Since $\tilde{\pi} \in D$, $\tilde{\pi} \neq \sigma^i$ so, $\tilde{\pi} = \pi$. This implies that $L^{sd}(\overset{\mathrm{T}}{P}_i, \pi) \supseteq L^{sd}(P_i, \pi) \setminus D$: Otherwise, there is $\sigma^i \in \Delta(A)$ such that $\sigma^i \in L^{sd}(P_i, \pi) \setminus (D \cup L^{sd}(\overset{\mathrm{T}}{P}_i, \pi))$. Let $s_i' \equiv (P, \pi, \sigma^i, n)$. Then, $h(s_i', s_{-i}) = \sigma^i$. So $h(s_i', s_{-i}) = \sigma^i \notin L^{sd}(\overset{\mathrm{T}}{P}_i, \pi) = L^{sd}(\overset{\mathrm{T}}{P}_i, h(s))$. This contradicts $s \in SNE^{sd}(\Gamma, \overset{\mathrm{T}}{P})$. Since $\pi \in \varphi(P)$ and for each $k \in N$, $L^{sd}(\overset{\mathrm{T}}{P}_k, \pi) \supseteq L^{sd}(P_k, \pi)$, by *invariance to lower SD monotonic transformations*, $\pi \in \varphi(\overset{\mathrm{T}}{P})$.

Case 3: There is a triple $i, j, k \in N$ such that $s_i \neq s_j, s_i \neq s_k$, and $s_j \neq s_k$. Recall that $\pi \equiv h(s)$.

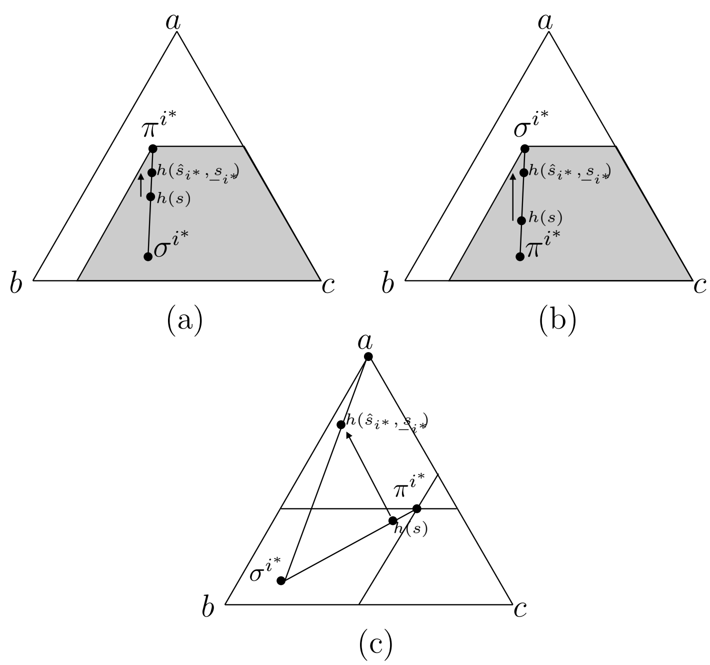

Figure 2: **Illustration of the proof of Proposition 2.** For each sub-case of 3.1, we see that $h(\hat{s}_{i^*}, s_{-i^*})\ P_{i^*}^{sd}\ h(s)$: (a) corresponds to the case where $\pi^{i^*}\ P_{i^*}^{sd}\ \sigma^{i^*}$; (b) corresponds to $\sigma^{i^*}\ P_{i^*}^{sd}\ \pi^{i^*}$; and (c) corresponds to the remaining case.

Case 3.1: $\pi = \frac{1}{n^{i^*}+1}\sigma^{i^*} + \frac{n^{i^*}}{n^{i^*}+1}\pi^{i^*}$ where $i^* \equiv \min\left\{\arg\max_{j \in N} n^j\right\}$.

(Figure 2 (a)) If $\pi^{i^*}\ \overset{\mathrm{T}}{P}{}_{i^*}^{sd}\ \sigma^{i^*}$, then let $\hat{s}_{i^*} \equiv (P^{i^*}, \pi^{i^*}, \sigma^{i^*}, n^{i^*} + 1)$.

(Figure 2 (b)) If $\sigma^{i^*}\ \overset{\mathrm{T}}{P}{}^{sd}{}_{i^*}\ \pi^{i^*}$, then let $\hat{s}_{i^*} \equiv (P^{i^*}, \sigma^{i^*}, \pi^{i^*}, n^{i^*} + 1)$. Note that σ^{i^*} and π^{i^*} are interchanged.

(Figure 2 (c)) Otherwise, let $\hat{s}_{i^*} \equiv (P^{i^*}, \hat{\pi}^{i^*}, \sigma^{i^*}, \hat{n}^{i^*})$, where $\hat{\pi}^{i^*} = \delta^{\tau(\overset{\mathrm{T}}{P}_{i^*})}$ and $\hat{n}^{i^*}$ is large enough that $\hat{n}^{i^*} > n^{i^*}$ and $\frac{1}{\hat{n}^{i^*}+1}\sigma^{i^*} + \frac{\hat{n}^{i^*}}{\hat{n}^{i^*}+1}\hat{\pi}^{i^*}\ \overset{\mathrm{T}}{P}{}_{i^*}^{sd}\ \pi$.

For each of the above cases, $h(\hat{s}_{i^*}, s_{-i^*})\ \overset{\mathrm{T}}{P}{}_{i^*}^{sd}\ \pi = h(s)$. This contradicts $s \in SNE^{sd}(\Gamma, \overset{\mathrm{T}}{P})$.

Case 3.2: $\pi = \bar{\pi}$. Let $\hat{s}_{i^*} \equiv (\hat{P}^{i^*}, \hat{\pi}^{i^*}, \hat{\sigma}^{i^*}, \hat{n}^{i^*}) \in S_{i^*}$ be such that $\hat{\pi}^{i^*} = \delta^{\tau(P_{i^*})}$, $\hat{\sigma}^{i^*}(\neq \hat{\pi}^{i^*})\ \overset{\mathrm{T}}{P^{sd}_{i^*}}\ \bar{\pi}$, and $\hat{n}^{i^*} \equiv \max_{l \in N} n^l + 2$. Then, $h(\hat{s}_{i^*}, s_{-i^*}) = \frac{1}{\hat{n}^{i^*}}\hat{\sigma}^{i^*} + \frac{\hat{n}^{i^*}}{\hat{n}^{i^*}+1}\hat{\pi}^{i^*}\ \overset{\mathrm{T}}{P^{sd}_{i^*}}\ \bar{\pi} = h(s)$. This contradicts $s \in SNE^{sd}(\Gamma, \overset{\mathrm{T}}{P})$.

We omit the proofs of parts (ii) and (iii) of Proposition 2. To prove (ii), we simply adapt the proof of (i) by replacing "$\sigma^i \in L^{sd}(P_i, \pi) \setminus D$" in the definition of h with "$\sigma^i \notin U^{sd}(P_i, \pi) \cup D$." To prove (iii), we replace it with "$\sigma^i \in L^{eu}(u_i, \pi) \setminus D$."

B.1 The proofs of (ii) and (iii) of Proposition 2
For referees. Not for publication.

We present the complete proofs with the changes mentioned above.

Implementation in weak SDNE

(⇒) Suppose that φ is *implementable in weak SDNE*. Suppose that it is implemented by $\Gamma \equiv (S, h)$. Let $P \in \mathcal{P}^N$ and $\pi \in \varphi(P)$. Then, there is $s \in WNE^{sd}(\Gamma, P)$ such that $h(s) = \pi$. Since $s \in WNE^{sd}(\Gamma, P)$, for each $i \in N$ and each $s'_i \in S_i$, $h(s'_i, s_{-i}) \notin U^{sd}(P_i, \pi) \setminus \{\pi\}$.

Let $P' \in \mathcal{P}^N$ be such that for each $i \in N$, $U^{sd}(P'_i, \pi) \subseteq U^{sd}(P_i, \pi)$. Then, for each $s'_i \in S_i$, $h(s'_i, s_{-i}) \notin U^{sd}(P'_i, \pi) \setminus \{\pi\}$. Thus, $s \in WNE^{sd}(\Gamma, P')$ so $\pi = h(s) \in \varphi(P')$.

(⇐) We define a canonical game form and show that it *implements φ in weak SDNE.*

For each $i \in N$, let $S_i \equiv \mathcal{P}^N \times \Delta(A) \times \Delta(A) \times \mathbb{N}_+$. Denote a generic member of S_i by $s_i \equiv (P^i, \pi^i, \sigma^i, n^i)$. Define $h : S \to \Delta(A)$ by setting, for each $s \in S$,

$$h(s) \equiv \begin{cases} \pi & \text{if for each } i \in N,\ s_i = (P, \pi, \sigma, n) \text{ and } \pi \in \varphi(P) \\ \sigma^i & \left\{\begin{array}{l} \text{if there is } i \in N \text{ such that} \\ \text{for each } j \in N \setminus \{i\}, s_j = (P, \pi, \sigma, n) \neq s_i = (P^i, \pi^i, \sigma^i, n^i) \\ \text{and } \sigma^i \notin U^{sd}(P_i, \pi) \cup D, \end{array}\right. \\ \pi & \left\{\begin{array}{l} \text{if there is } i \in N \text{ such that} \\ \text{for each } j \in N \setminus \{i\}, s_j = (P, \pi, \sigma, n) \neq s_i = (P^i, \pi^i, \sigma^i, n^i) \\ \text{and } \sigma^i \in U^{sd}(P_i, \pi) \cup D, \end{array}\right. \\ \frac{1}{n^{i^*}+1}\sigma^{i^*} + \frac{n^{i^*}}{n^{i^*}+1}\pi^{i^*} & \left\{\begin{array}{l} \text{if there is a triple } i, j, k \in N \text{ such that } s_i \neq s_j, s_i \neq s_k, s_j \neq s_k, \\ \text{and } \pi^{i^*} \neq \sigma^{i^*}, \text{ where } i^* \equiv \min\left\{\arg\max_{j \in N} n^j\right\}, \text{ and} \end{array}\right. \\ \bar{\pi} & \text{otherwise.} \end{cases}$$

Let $\Gamma \equiv (S, h)$. Our goal is to show that for each $P \in \mathcal{P}^N$, $h(WNE^{sd}(\Gamma, P) = \varphi(P)$.

First, we show that $\varphi(P) \subseteq h(WNE^{sd}(\Gamma, P))$. Let $\pi \in \varphi(P)$. Let $s \in S$ be such that for each $i \in N$, $s_i = (P, \pi, \pi, 0)$. For each $i \in N$ and each $s'_i \in S_i \setminus \{s_i\}$, either $h(s'_i, s_{-i}) = h(s) = \pi$ or $h(s'_i, s_{-i}) \notin U^{sd}(P_i, \pi) \cup D$. Thus, $s \in WNE^{sd}(\Gamma, P)$. Consequently $\pi = h(s) \in h(WNE^{sd}(\Gamma, P))$.

Next, consider $\overset{T}{P} \in \mathcal{P}^N$ and $s \in WNE^{sd}(\Gamma, \overset{T}{P})$. We complete the proof by showing that $h(s) \in \varphi(\overset{T}{P})$. Let $\pi \equiv h(s)$.

Recall that for each $P_0 \in \mathcal{P}$ and each $\pi \in \Delta(A)$, $U^{sd}(P_0, \pi)$ is closed and convex. Thus, for each $P_0' \in \mathcal{P}$ if $U^{sd}(P_0', \pi) \subseteq U^{sd}(P_0, \pi) \cup D$ then $U^{sd}(P_0', \pi) \subseteq U^{sd}(P_0, \pi)$.

Case 1: For each $i \in N$, $s_i = (P, \pi, \sigma, n)$ and $\pi \in \varphi(P)$. Then, for each $i \in N$, $U^{sd}(\overset{T}{P}_i, \pi) \subseteq U^{sd}(P_i, \pi) \cup D$: otherwise, there is $\sigma^i \in \Delta(A)$ such that $\sigma^i \neq \pi$ and $\sigma^i \in U^{sd}(\overset{T}{P}_i, \pi) \setminus (D \cup U^{sd}(P_i, \pi))$. Let $s_i' \equiv (P, \pi, \sigma^i, n)$. Then $h(s_i', s_{-i}) = \sigma^i$. So $h(s_i', s_{-i}) = \sigma^i \in U^{sd}(\overset{T}{P}_i, \pi) \setminus \{\pi\} = U^{sd}(\overset{T}{P}_i, h(s)) \setminus \{\pi\}$. This contradicts $s \in WNE^{sd}(\Gamma, \overset{T}{P})$. Since $\pi \in \varphi(P)$ and for each $i \in N$, $U^{sd}(\overset{T}{P}_i, \pi) \subseteq U^{sd}(P_i, \pi)$, by *invariance to upper SD monotonic transformations*, $\pi \in \varphi(\overset{T}{P})$.

Case 2: There is $i \in N$ such that, for each $j \in N \setminus \{i\}, s_j = (P, \pi, \sigma, n) \neq s_i$. Let $\tilde{\pi} \equiv h(s)$. Then for each $j \in N \setminus \{i\}$, $\tilde{\pi} = \delta^{\tau(\overset{T}{P}_j)}$: otherwise, let $\hat{s}_j \in S_j$ be such that $\hat{n}^j \equiv \max\{n^i, n\} + 1$, $\hat{\sigma}^j \equiv \tilde{\pi}$, $\hat{P}^j \in \mathcal{P}^N$, and $\hat{\pi}^j = \delta^{\tau(\overset{T}{P}_j)}$. Then, $h(\hat{s}_j, s_{-j}) = \frac{1}{\hat{n}^j+1}\tilde{\pi} + \frac{\hat{n}^j}{\hat{n}^j+1}\delta^{\tau(\overset{T}{P}_j)} \; \overset{T}{P}{}^{sd}_j \; \tilde{\pi}$. This contradicts $s \in WNE^{sd}(\Gamma, \overset{T}{P})$. Thus, for each $j \in N \setminus \{i\}$, $\tilde{\pi} = \delta^{\tau(\overset{T}{P}_j)} \in D$.

Since $\tilde{\pi} \in D$, $\tilde{\pi} \neq \sigma^i$ so, $\tilde{\pi} = \pi$. This implies that $U^{sd}(\overset{T}{P}_i, \pi) \subseteq U^{sd}(P_i, \pi) \cup D$: Otherwise, there is $\sigma^i \in \Delta(A)$ such that $\sigma^i \neq \pi$ and $\sigma^i \in U^{sd}(\overset{T}{P}_i, \pi) \setminus (D \cup U^{sd}(P_i, \pi))$. Let $s_i' \equiv (P, \pi, \sigma^i, n)$. Then, $h(s_i', s_{-i}) = \sigma^i$. So $h(s_i', s_{-i}) = \sigma^i \in U^{sd}(\overset{T}{P}_i, \pi) = U^{sd}(\overset{T}{P}_i, h(s))$. This contradicts $s \in WNE^{sd}(\Gamma, \overset{T}{P})$. Since $\pi \in \varphi(P)$ and for each $k \in N$, $U^{sd}(\overset{T}{P}_k, \pi) \subseteq U^{sd}(P_k, \pi)$, by *invariance to upper SD monotonic transformations*, $\pi \in \varphi(\overset{T}{P})$.

Case 3: There is a triple $i, j, k \in N$ such that $s_i \neq s_j, s_i \neq s_k$, and $s_j \neq s_k$. Recall that $\pi \equiv h(s)$.

Case 3.1: $\pi = \frac{1}{n^{i^*}+1}\sigma^{i^*} + \frac{n^{i^*}}{n^{i^*}+1}\pi^{i^*}$ where $i^* \equiv \min\left\{\arg\max_{j \in N} n^j\right\}$.

If $\pi^{i^*} \; \overset{T}{P}{}^{sd}_{i^*} \; \sigma^{i^*}$, then let $\hat{s}_{i^*} \equiv (P^{i^*}, \pi^{i^*}, \sigma^{i^*}, n^{i^*} + 1)$.

If $\sigma^{i^*} \; \overset{T}{P}{}^{sd}_{i^*} \; \pi^{i^*}$, then let $\hat{s}_{i^*} \equiv (P^{i^*}, \sigma^{i^*}, \pi^{i^*}, n^{i^*} + 1)$. Note that σ^{i^*} and π^{i^*} are interchanged.

Otherwise, let $\hat{s}_{i^*} \equiv (P^{i^*}, \hat{\pi}^{i^*}, \sigma^{i^*}, \hat{n}^{i^*})$, where $\hat{\pi}^{i^*} = \delta^{\tau(\overset{T}{P}_{i^*})}$ and $\hat{n}^{i^*}$ is large enough that $\hat{n}^{i^*} > n^{i^*}$ and $\frac{1}{\hat{n}^{i^*}+1}\sigma^{i^*} + \frac{\hat{n}^{i^*}}{\hat{n}^{i^*}+1}\hat{\pi}^{i^*} \; \overset{T}{P}{}^{sd}_{i^*} \; \pi$.

For each of the above cases, $h(\hat{s}_{i^*}, s_{-i^*}) \; \overset{T}{P}{}^{sd}_{i^*} \; \pi = h(s)$. This contradicts $s \in WNE^{sd}(\Gamma, \overset{T}{P})$.

Case 3.2: $\pi = \bar{\pi}$. Let $\hat{s}_{i^*} \equiv (\hat{P}^{i^*}, \hat{\pi}^{i^*}, \hat{\sigma}^{i^*}, \hat{n}^{i^*}) \in S_{i^*}$ be such that $\hat{\pi}^{i^*} = \delta^{\tau(P_{i^*})}$, $\hat{\sigma}^{i^*} (\neq \hat{\pi}^{i^*}) \; \overset{T}{P}{}^{sd}_{i^*} \; \bar{\pi}$, and $\hat{n}^{i^*} \equiv \max_{l \in N} n^l + 2$. Then, $h(\hat{s}_{i^*}, s_{-i^*}) = \frac{1}{\hat{n}^{i^*}}\hat{\sigma}^{i^*} + \frac{\hat{n}^{i^*}}{\hat{n}^{i^*}+1}\hat{\pi}^{i^*} \; \overset{T}{P}{}^{sd}_{i^*} \; \bar{\pi} = h(s)$. This contradicts $s \in WNE^{sd}(\Gamma, \overset{T}{P})$.

Implementation in expected utility NE

(⇒) Suppose that φ is *implementable in expected utility NE*. Suppose that it is implemented by $\Gamma \equiv (S, h)$. Let $u \in \mathcal{U}^N$ and $\pi \in \varphi \circ \rho(u)$. Then, there is $s \in NE(\Gamma, u)$ such that $h(s) = \pi$. Since $s \in NE(\Gamma, u)$, for each $i \in N$ and each $s'_i \in S_i$, $h(s'_i, s_{-i}) \in L^{eu}(u_i, \pi)$.

Let $u' \in \mathcal{U}^N$ be such that for each $i \in N$, $L^{eu}(u'_i, \pi) \supseteq L^{eu}(u_i, \pi)$. Then, for each $s'_i \in S_i$, $h(s'_i, s_{-i}) \in L^{eu}(u'_i, \pi)$. Thus, $s \in NE(\Gamma, u)$ so $\pi = h(s) \in \varphi \circ \rho(u')$.

(⇐) We define a canonical game form and show that it *implements* φ *in expected utility NE*.

For each $i \in N$, let $S_i \equiv \mathcal{U}^N \times \Delta(A) \times \Delta(A) \times \mathbb{N}_+$. Denote a generic member of S_i by $s_i \equiv (u^i, \pi^i, \sigma^i, n^i)$. Define $h : S \to \Delta(A)$ by setting, for each $s \in S$,

$$h(s) \equiv \begin{cases} \pi & \text{if for each } i \in N,\ s_i = (P, \pi, \sigma, n) \text{ and } \pi \in \varphi \circ \rho(u) \\[2ex] \sigma^i & \left\{\begin{array}{l} \text{if there is } i \in N \text{ such that} \\ \text{for each } j \in N \setminus \{i\}, s_j = (u, \pi, \sigma, n) \neq s_i = (u^i, \pi^i, \sigma^i, n^i) \\ \text{and } \sigma^i \in L^{eu}(u_i, \pi) \setminus D, \end{array}\right. \\[4ex] \pi & \left\{\begin{array}{l} \text{if there is } i \in N \text{ such that} \\ \text{for each } j \in N \setminus \{i\}, s_j = (u, \pi, \sigma, n) \neq s_i = (u^i, \pi^i, \sigma^i, n^i) \\ \text{and } \sigma^i \notin L^{eu}(u_i, \pi) \setminus D, \end{array}\right. \\[4ex] \frac{1}{n^{i^*}+1}\sigma^{i^*} + \frac{n^{i^*}}{n^{i^*}+1}\pi^{i^*} & \left\{\begin{array}{l} \text{if there is a triple } i, j, k \in N \text{ such that } s_i \neq s_j, s_i \neq s_k, s_j \neq s_k, \\ \text{and } \pi^{i^*} \neq \sigma^{i^*}, \text{ where } i^* \equiv \min\left\{\arg\max\limits_{j \in N} n^j\right\}, \text{ and} \end{array}\right. \\[4ex] \bar{\pi} & \text{otherwise.} \end{cases}$$

Let $\Gamma \equiv (S, h)$. Our goal is to show that for each for each $u \in \mathcal{U}^N$, $h(NE(\Gamma, u)) = \varphi \circ \rho(u)$.

First, we show that $\varphi \circ \rho(u) \subseteq h(NE(\Gamma, u))$. Let $\pi \in \varphi \circ \rho(u)$. Let $s \in S$ be such that for each $i \in N$, $s_i = (u, \pi, \pi, 0)$. For each $i \in N$ and each $s'_i \in S_i \setminus \{s_i\}$, either $h(s'_i, s_{-i}) = h(s) = \pi$ or $h(s'_i, s_{-i}) \in L^{eu}(u_i, \pi) \setminus D$. Thus, $s \in NE(\Gamma, u)$. Consequently $\pi = h(s) \in h(NE(\Gamma, u))$.

Next, consider $\overset{\mathrm{T}}{u} \in \mathcal{U}^N$ and $s \in NE(\Gamma, \overset{\mathrm{T}}{u})$. We complete the proof by showing that $h(s) \in \varphi \circ \rho(\overset{\mathrm{T}}{u})$. Let $\pi \equiv h(s)$,

Recall that for each $u_0 \in \mathcal{U}$ and each $\pi \in \Delta(A)$, $L^{eu}(u_0, \pi)$ is closed and convex. Thus, for each $u'_0 \in \mathcal{U}$ if $L^{eu}(u'_0, \pi) \supseteq L^{eu}(u_0, \pi) \setminus D$ then $L^{eu}(u'_0, \pi) \supseteq L^{eu}(u_0, \pi)$.

Case 1: For each $i \in N$, $s_i = (u, \pi, \sigma, n)$ and $\pi \in \varphi \circ \rho(u)$. Then, for each $i \in N$, $L^{eu}(\overset{\mathrm{T}}{u}_i, \pi) \supseteq L^{eu}(u_i, \pi) \setminus D$: otherwise, there is $\sigma^i \in \Delta(A)$ such that $\sigma^i \in L^{eu}(u_i, \pi) \setminus (D \cup L^{eu}(\overset{\mathrm{T}}{u}_i, \pi))$. Let $s'_i \equiv (u, \pi, \sigma^i, n)$. Then $h(s'_i, s_{-i}) = \sigma^i$. So $h(s'_i, s_{-i}) = \sigma^i$ where $\overset{\mathrm{T}}{U}_i(\sigma^i) > \overset{\mathrm{T}}{U}_i(\pi)$. This contradicts $s \in NE(\Gamma, \overset{\mathrm{T}}{u})$. Since $\pi \in \varphi \circ \rho(u)$ and for each $i \in N$, $L^{eu}(\overset{\mathrm{T}}{u}_i, \pi) \supseteq L^{eu}(u_i, \pi)$, by *invariance to Maskin monotonic transformations*, $\pi \in \varphi \circ \rho(\overset{\mathrm{T}}{u})$.

Case 2: There is $i \in N$ such that, for each $j \in N \setminus \{i\}$, $s_j = (u, \pi, \sigma, n) \neq s_i$. Let $\tilde{\pi} \equiv h(s)$. Then for each $j \in N \setminus \{i\}$, $\tilde{\pi} = \delta^{\tau(\overset{\mathrm{T}}{u}_j)}$:[25] otherwise, let $\hat{s}_j \in S_j$ be such that $\hat{n}^j \equiv \max\{n^i, n\}+1$, $\hat{\sigma}^j \equiv \tilde{\pi}$,

[25]For $u_0 \in \mathcal{U}$, we write $\tau(u_0)$ to mean $\tau(\rho(u_0))$.

$\hat{u}^j \in \mathcal{U}^N$, and $\hat{\pi}^j = \delta^{\tau(\overset{\mathrm{T}}{u}_j)}$. Then, since $h(\hat{s}_j, s_{-j}) = \frac{1}{\hat{n}^j+1}\tilde{\pi} + \frac{\hat{n}^j}{\hat{n}^j+1}\delta^{\tau(\overset{\mathrm{T}}{u}_j)}$, $\overset{\mathrm{T}}{U}_j(h(\hat{s}_j, s_{-j})) > \overset{\mathrm{T}}{U}_j(\tilde{\pi})$. This contradicts $s \in NE(\Gamma, \overset{\mathrm{T}}{u})$. Thus, for each $j \in N \setminus \{i\}$, $\tilde{\pi} = \delta^{\tau(\overset{\mathrm{T}}{u}_j)} \in D$.

Since $\tilde{\pi} \in D$, $\tilde{\pi} \neq \sigma^i$ so, $\tilde{\pi} = \pi$. This implies that $L^{eu}(\overset{\mathrm{T}}{u}_i, \pi) \supseteq L^{eu}(u_i, \pi) \setminus D$: Otherwise, there is $\sigma^i \in \Delta(A)$ such that $\sigma^i \in L^{eu}(u_i, \pi) \setminus (D \cup L^{eu}(\overset{\mathrm{T}}{u}_i, \pi)))$. Let $s_i' \equiv (u, \pi, \sigma^i, n)$. Then, $h(s_i', s_{-i}) = \sigma^i$ where $\overset{\mathrm{T}}{U}_i(\sigma^i) > \overset{\mathrm{T}}{U}_i(\pi)$. This contradicts $s \in NE(\Gamma, \overset{\mathrm{T}}{u})$. Since $\pi \in \varphi \circ \rho(u)$ and for each $k \in N$, $L^{eu}(\overset{\mathrm{T}}{u}_k, \pi) \supseteq L^{eu}(u_k, \pi)$, by *invariance to Maskin monotonic transformations*, $\pi \in \varphi \circ \rho(\overset{\mathrm{T}}{u})$.

Case 3: There is a triple $i, j, k \in N$ such that $s_i \neq s_j, s_i \neq s_k$, and $s_j \neq s_k$. Recall that $\pi \equiv h(s)$.

Case 3.1: $\pi = \frac{1}{n^{i^*}+1}\sigma^{i^*} + \frac{n^{i^*}}{n^{i^*}+1}\pi^{i^*}$ where $i^* \equiv \min\left\{\arg\max_{j \in N} n^j\right\}$.

If $\overset{\mathrm{T}}{U}_{i^*}(\pi^{i^*}) > \overset{\mathrm{T}}{U}_{i^*}(\sigma^{i^*})$, then let $\hat{s}_{i^*} \equiv (u^{i^*}, \pi^{i^*}, \sigma^{i^*}, n^{i^*} + 1)$.

If $\overset{\mathrm{T}}{U}_{i^*}(\sigma^{i^*}) > \overset{\mathrm{T}}{U}_{i^*}(\pi^{i^*})$, then let $\hat{s}_{i^*} \equiv (u^{i^*}, \sigma^{i^*}, \pi^{i^*}, n^{i^*} + 1)$. Note that σ^{i^*} and π^{i^*} are interchanged.

Otherwise, let $\hat{s}_{i^*} \equiv (u^{i^*}, \hat{\pi}^{i^*}, \sigma^{i^*}, \hat{n}^{i^*})$, where $\hat{\pi}^{i^*} = \delta^{\tau(\overset{\mathrm{T}}{u}_{i^*})}$ and $\hat{n}^{i^*}$ is large enough that $\hat{n}^{i^*} > n^{i^*}$ and $\overset{\mathrm{T}}{U}_{i^*}\left(\frac{1}{\hat{n}^{i^*}+1}\sigma^{i^*} + \frac{\hat{n}^{i^*}}{\hat{n}^{i^*}+1}\hat{\pi}^{i^*}\right) > \overset{\mathrm{T}}{U}_{i^*}(\pi)$.

For each of the above cases, $\overset{\mathrm{T}}{U}_{i^*}(h(\hat{s}_{i^*}, s_{-i^*})) > \overset{\mathrm{T}}{U}_{i^*}(\pi) = \overset{\mathrm{T}}{U}_{i^*}(h(s))$. This contradicts $s \in NE(\Gamma, \overset{\mathrm{T}}{u})$.

Case 3.2: $\pi = \bar{\pi}$. Let $\hat{s}_{i^*} \equiv (\hat{u}^{i^*}, \hat{\pi}^{i^*}, \hat{\sigma}^{i^*}, \hat{n}^{i^*}) \in S_{i^*}$ be such that $\hat{\pi}^{i^*} = \delta^{\tau(u_{i^*})}$, $\hat{\sigma}^{i^*}(\neq \hat{\pi}^{i^*}) \notin L^{eu}(\overset{\mathrm{T}}{u}_{i^*}, \bar{\pi})$, and $\hat{n}^{i^*} \equiv \max_{l \in N} n^l + 2$. Then, since $h(\hat{s}_{i^*}, s_{-i^*}) = \frac{1}{\hat{n}^{i^*}}\hat{\sigma}^{i^*} + \frac{\hat{n}^{i^*}}{\hat{n}^{i^*}+1}\hat{\pi}^{i^*}$, $\overset{\mathrm{T}}{U}_{i^*}(h(\hat{s}_{i^*}, s_{-i^*})) > \overset{\mathrm{T}}{U}_{i^*}(\bar{\pi}) = \overset{\mathrm{T}}{U}_{i^*}(h(s))$. This contradicts $s \in NE(\Gamma, \overset{\mathrm{T}}{u})$.

C Proof of Proposition 4

($\bigcup \boldsymbol{\Phi}$) Let $P \in \mathcal{P}^N$ and $\pi \in (\bigcup \Phi)(P)$. Then, there is $\phi \in \Phi$ such that $\pi \in \phi(P)$. Let $P' \in \mathcal{P}^N$ be a *support monotonic transformation of P at π*. By *invariance to support monotonic transformations*, $\pi \in \phi(P')$. Thus, $\pi \in (\bigcup \Phi)(P')$.

($\bigcap \boldsymbol{\Phi}$) Let $P \in \mathcal{P}^N$ and $\pi \in (\bigcap \Phi)(P)$. Then, for each $\phi \in \Phi$, $\pi \in \phi(P)$. Let $P' \in \mathcal{P}^N$ be a *support monotonic transformation of P at π*. By *invariance to support monotonic transformations*, for each $\phi \in \Phi$, $\pi \in \phi(P')$. Thus, $\pi \in (\bigcap \Phi)(P')$.

(**Convex combination**) Let $P \in \mathcal{P}^N$, $\alpha \in (0, 1)$, $\underline{\pi} \in \phi(P)$, and $\overline{\pi} \in \varphi(P)$. Let $\pi \equiv (\alpha\underline{\pi} + (1-\alpha)\overline{\pi}) \in (\alpha\phi + (1-\alpha)\varphi)(P)$. Let $P' \in \mathcal{P}^N$ be a *support monotonic transformation of P at π*. For each $i \in N$ and each $a \in A$, $\pi_i(a) = 0$ if and only if $\underline{\pi}_i(a) = \overline{\pi}_i(a) = 0$. Thus, P' is a *support monotonic transformation of P at both $\underline{\pi}$ and $\overline{\pi}$*. By *invariance to support monotonic transformations*, $\underline{\pi} \in \phi(P')$ and $\overline{\pi} \in \varphi(P')$. Thus, $\pi \in (\alpha\phi + (1-\alpha)\varphi)(P')$.

D Proof of Proposition 2 adapted for economic environments

For referees. Not for publication.

Again we only show the proof of (i).

($\Rightarrow$) Suppose that φ is *implementable in strong SDNE*. Suppose that it is implemented by $\Gamma \equiv (S, h)$. Let $P \in \mathcal{P}^N$ and $\pi \in \varphi(P)$. Then, there is $s \in SNE^{sd}(\Gamma, P)$ such that $h(s) = \pi$. Since $s \in SNE^{sd}(\Gamma, P)$, for each $i \in N$ and each $s'_i \in S_i$, $h_i(s'_i, s_{-i}) \in L^{sd}(P_i, \pi_i)$.

Let $P' \in \mathcal{P}^N$ be such that for each $i \in N$, $L^{sd}(P'_i, \pi_i) \supseteq L^{sd}(P_i, \pi_i)$. Then, for each $s'_i \in S_i$, $h(s'_i, s_{-i}) \in L^{sd}(P'_i, \pi_i)$. Thus, $s \in SNE^{sd}(\Gamma, P')$ so $\pi = h(s) \in \varphi(P')$.

($\Leftarrow$) In this part of the proof we define a game form and show that it *implements φ in strong SDNE*.

First, we present some notation. For each $i \in N$ and each $P_i \in \mathcal{P}_i$ and each $Y_i \subseteq X_i$, let $\boldsymbol{\tau(P_i, Y_i)}$ be the **top bundle under $\boldsymbol{P_i}$ in $\boldsymbol{Y_i}$**. That is, $\tau(P_i, Y_i) = x_i$ if and only if $x_i \in Y_i$ and for each $y_i \in Y_i \setminus \{x_i\}$, $x_i \; P_i \; y_i$. For convenience, we usually drop X_i in the expression $\tau(P_i, X_i)$ and write $\tau(P_i)$ instead. That is, when the second argument is omitted, it is X_i. For each $x_i \in X_i$, let $\boldsymbol{\delta^{x_i}} \in \Delta(X_i)$ be such that $\delta^{x_i}(x_i) = 1$. Let $\boldsymbol{D_i} \equiv \{\delta^{x_i} : x_i \in X_i\}$. Let $\boldsymbol{\bar{\pi}} \in \Delta(F)$ be such that for each $x \in F$, $\bar{\pi}(x) \equiv \frac{1}{|F|}$.

We next define a canonical game form. For each $i \in N$, let $S_i \equiv \mathcal{P}^N \times \Delta(F) \times \Delta(F) \times \mathbb{N}_+$. Denote a generic member of S_i by $s_i \equiv (P^i, \pi^i, \sigma^i, n^i)$. Define $h : S \to \Delta(A)$ by setting, for each $s \in S$,

$$h(s) \equiv \begin{cases} \pi & \text{if for each } i \in N,\ s_i = (P, \pi, \sigma, n) \text{ and } \pi \in \varphi(P) \\[2ex] \sigma^i & \left\{\begin{array}{l} \text{if there is } i \in N \text{ such that} \\ \text{for each } j \in N \setminus \{i\}, s_j = (P, \pi, \sigma, n) \neq s_i = (P^i, \pi^i, \sigma^i, n^i) \\ \text{and } \sigma^i_i \in L^{sd}(P_i, \pi_i) \setminus D_i, \end{array}\right. \\[2ex] \pi & \left\{\begin{array}{l} \text{if there is } i \in N \text{ such that} \\ \text{for each } j \in N \setminus \{i\}, s_j = (P, \pi, \sigma, n) \neq s_i = (P^i, \pi^i, \sigma^i, n^i) \\ \text{and } \sigma^i_i \notin L^{sd}(P_i, \pi_i) \setminus D_i, \end{array}\right. \\[2ex] \frac{1}{n^{i^*}+1}\sigma^{i^*} + \frac{n^{i^*}}{n^{i^*}+1}\pi^{i^*} & \left\{\begin{array}{l} \text{if there is a triple } i, j, k \in N \text{ such that } s_i \neq s_j, s_i \neq s_k, s_j \neq s_k, \\ \text{and } \pi^{i^*}_{i^*} \neq \sigma^{i^*}_{i^*}, \text{ where } i^* \equiv \min\left\{\arg\max_{j \in N} n^j\right\}, \text{ and} \end{array}\right. \\[2ex] \bar{\pi} & \text{otherwise.} \end{cases}$$

Let $\Gamma \equiv (S, h)$. Our goal is to show that for each $P \in \mathcal{P}^N$, $h(SNE^{sd}(\Gamma, P)) = \varphi(P)$.

First, we show that $\varphi(P) \subseteq h(SNE^{sd}(\Gamma, P))$. Let $\pi \in \varphi(P)$. Let $s \in S$ be such that for each $i \in N$, $s_i = (P, \pi, \pi, 0)$. For each $i \in N$ and each $s'_i \in S_i \setminus \{s_i\}$, either $h_i(s'_i, s_{-i}) = h_i(s) = \pi_i$ or $h_i(s'_i, s_{-i}) \in L^{sd}(P_i, \pi_i) \setminus D_i$. Thus, $s \in SNE^{sd}(\Gamma, P)$. Consequently $\pi = h(s) \in h(SNE^{sd}(\Gamma, P))$.

Next, consider $\overset{\mathrm{T}}{P} \in \mathcal{P}^N$ and $s \in SNE^{sd}(\Gamma, \overset{\mathrm{T}}{P})$. We complete the proof by showing that $h(s) \in \varphi(\overset{\mathrm{T}}{P})$. Let $\pi \equiv h(s)$.

Recall that for each $i \in N$, each $P_i \in \mathcal{P}_i$, and each $\pi_i \in \Delta(X_i)$, $L^{sd}(P_i, \pi_i)$ is closed. Thus, for each $P'_i \in \mathcal{P}_i$ if $L^{sd}(P'_i, \pi_i) \supseteq L^{sd}(P_i, \pi_i) \setminus D_i$ then $L^{sd}(P'_i, \pi_i) \supseteq L^{sd}(P_i, \pi_i)$.

Case 1: For each $i \in N$, $s_i = (P, \pi, \sigma, n)$ and $\pi \in \varphi(P)$. Then, for each $i \in N$, $L^{sd}(\overset{\mathrm{T}}{P}_i, \pi_i) \supseteq L^{sd}(P_i, \pi_i) \setminus D_i$: Otherwise, there is $\sigma^i \in \Delta(F)$ such that $\sigma^i_i \in L^{sd}(P_i, \pi_i) \setminus (D_i \cup L^{sd}(\overset{\mathrm{T}}{P}_i, \pi_i))$. Let $s'_i \equiv (P, \pi, \sigma^i, n)$. Then $h_i(s'_i, s_{-i}) = \sigma^i_i$. So $h_i(s'_i, s_{-i}) = \sigma^i_i \notin L^{sd}(\overset{\mathrm{T}}{P}_i, \pi_i) = L^{sd}(\overset{\mathrm{T}}{P}_i, h_i(s))$. This contradicts $s \in SNE^{sd}(\Gamma, \overset{\mathrm{T}}{P})$. Since $\pi \in \varphi(P)$ and for each $i \in N$, $L^{sd}(\overset{\mathrm{T}}{P}_i, \pi_i) \supseteq L^{sd}(P_i, \pi_i)$, by *invariance to lower SD monotonic transformations*, $\pi \in \varphi(\overset{\mathrm{T}}{P})$.

Case 2: There is $i \in N$ such that, for each $j \in N \setminus \{i\}$, $s_j = (P, \pi, \sigma, n) \neq s_i$. Let $\tilde{\pi} \equiv h(s)$. Then for each $j \in N \setminus \{i\}$, $\tilde{\pi}_j = \delta^{\tau(\overset{\mathrm{T}}{P}_j)}$: otherwise, let $\hat{s}_j \in S_j$ be such that $\hat{n}^j \equiv \max\{n^i, n\} + 1$, $\hat{\sigma}^j \equiv \tilde{\pi}$, $\hat{P}^j \in \mathcal{P}^N$, and $\hat{\pi}^j \in \Delta(F)$ be such that $\hat{\pi}^j_j = \delta^{\tau(\overset{\mathrm{T}}{P}_j)}$. Then, $h_j(\hat{s}_j, s_{-j}) = \frac{1}{\hat{n}^j+1}\tilde{\pi}_j + \frac{\hat{n}^j}{\hat{n}^j+1}\delta^{\tau(\overset{\mathrm{T}}{P}_j)} \; \overset{\mathrm{T}}{P}^{sd}_j \; \tilde{\pi}_j$. This contradicts $s \in SNE^{sd}(\Gamma, \overset{\mathrm{T}}{P})$. Thus, for each $j \in N \setminus \{i\}$, $\tilde{\pi}_j = \delta^{\tau(\overset{\mathrm{T}}{P}_j)}$.

Since $\tilde{\pi} \in \varphi(P)$ and for each $j \in N \setminus \{i\}$, $\tilde{\pi}_j = \delta^{\tau(\overset{\mathrm{T}}{P}_j)}$, by *regularity* of φ, $\tilde{\pi}_i \in D_i$. Thus, $\tilde{\pi} \neq \sigma^i$ so, $\tilde{\pi} = \pi$. This implies that $L^{sd}(\overset{\mathrm{T}}{P}_i, \pi_i) \supseteq L^{sd}(P_i, \pi_i) \setminus D_i$: Otherwise, there is $\sigma^i \in \Delta(F)$ such that $\sigma^i_i \in L^{sd}(P_i, \pi_i) \setminus (D_i \cup L^{sd}(\overset{\mathrm{T}}{P}_i, \pi))$. Let $s'_i \equiv (P, \pi, \sigma^i, n)$. Then, $h(s'_i, s_{-i}) = \sigma^i$. So $h_i(s'_i, s_{-i}) = \sigma^i_i \notin L^{sd}(\overset{\mathrm{T}}{P}_i, \pi_i) = L^{sd}(\overset{\mathrm{T}}{P}_i, h_i(s))$. This contradicts $s \in SNE^{sd}(\Gamma, \overset{\mathrm{T}}{P})$. Since $\pi \in \varphi(P)$ and for each $k \in N$, $L^{sd}(\overset{\mathrm{T}}{P}_k, \pi_k) \supseteq L^{sd}(P_k, \pi_k)$, by *invariance to lower SD monotonic transformations*, $\pi \in \varphi(\overset{\mathrm{T}}{P})$.

Case 3: There is a triple $i, j, k \in N$ such that $s_i \neq s_j, s_i \neq s_k$, and $s_j \neq s_k$. Recall that $\pi \equiv h(s)$.

Case 3.1: $\pi = \frac{1}{n^{i^*}+1}\sigma^{i^*} + \frac{n^{i^*}}{n^{i^*}+1}\pi^{i^*}$ where $i^* \equiv \min\left\{\arg\max_{j \in N} n^j\right\}$.

If $\pi^{i^*}_{i^*} \; \overset{\mathrm{T}}{P}^{sd}_{i^*} \; \sigma^{i^*}_{i^*}$, then let $\hat{s}_{i^*} \equiv (P^{i^*}, \pi^{i^*}, \sigma^{i^*}, n^{i^*} + 1)$.

If $\sigma^{i^*}_{i^*} \; \overset{\mathrm{T}}{P}^{sd}{}_{i^*} \; \pi^{i^*}_{i^*}$, then let $\hat{s}_{i^*} \equiv (P^{i^*}, \sigma^{i^*}, \pi^{i^*}, n^{i^*} + 1)$. Note that σ^{i^*} and π^{i^*} are interchanged.

Otherwise, let $\hat{s}_{i^*} \equiv (P^{i^*}, \hat{\pi}^{i^*}, \sigma^{i^*}, \hat{n}^{i^*})$, where $\hat{\pi}^{i^*}_{i^*} = \delta^{\tau(\overset{\mathrm{T}}{P}_{i^*})}$ and $\hat{n}^{i^*}$ is large enough that $\hat{n}^{i^*} > n^{i^*}$ and $\frac{1}{\hat{n}^{i^*}+1}\sigma^{i^*}_{i^*} + \frac{\hat{n}^{i^*}}{\hat{n}^{i^*}+1}\hat{\pi}^{i^*}_{i^*} \; \overset{\mathrm{T}}{P}^{sd}_{i^*} \; \pi_{i^*}$.

For each of the above cases, $h_{i^*}(\hat{s}_{i^*}, s_{-i^*}) \; \overset{\mathrm{T}}{P}^{sd}_{i^*} \; \pi_{i^*} = h_{i^*}(s)$. This contradicts $s \in SNE^{sd}(\Gamma, \overset{\mathrm{T}}{P})$.

Case 3.2: $\pi = \bar{\pi}$. Let $\hat{s}_{i^*} \equiv (\hat{P}^{i^*}, \hat{\pi}^{i^*}, \hat{\sigma}^{i^*}, \hat{n}^{i^*}) \in S_{i^*}$ be such that $\hat{\pi}^{i^*}_{i^*} = \delta^{\tau(P_{i^*})}$, $\hat{\sigma}^{i^*}_{i^*} (\neq \hat{\pi}^{i^*}_{i^*}) \; \overset{\mathrm{T}}{P}^{sd}_{i^*} \; \bar{\pi}_{i^*}$, and $\hat{n}^{i^*} \equiv \max_{l \in N} n^l + 2$. Then, $h_{i^*}(\hat{s}_{i^*}, s_{-i^*}) = \frac{1}{\hat{n}^{i^*}}\hat{\sigma}^{i^*}_{i^*} + \frac{\hat{n}^{i^*}}{\hat{n}^{i^*}+1}\hat{\pi}^{i^*}_{i^*} \; \overset{\mathrm{T}}{P}^{sd}_{i^*} \; \bar{\pi}_{i^*} = h_{i^*}(s)$. This contradicts $s \in SNE^{sd}(\Gamma, \overset{\mathrm{T}}{P})$.

E Implementability of other solutions

We list a few other solutions discussed in the literature. The following is less demanding than F^{sd} as it drops the requirement of SD comparability.

Weak SD no-envy solution, $\boldsymbol{WF^{sd}}$: For each $P \in \mathcal{P}^N$, $WF^{sd}(P) = \{\pi \in \Delta(F) :$ for each $i \in N$, there is no $j \in N$ such that $\pi_j \; P_i^{sd} \; \pi_i\}$.

Let $\theta : N \to \{1, \cdots, n\}$ be a one-to-one mapping from N to itself. Let Θ be the set of all such mappings. We define a class of single-valued solutions. Each member in this class is indexed by such an ordering and sequentially assigns each agent his most preferred object, among those remaining, with certainty.

Sequential priority solution associated with $\boldsymbol{\theta \in \Theta}$, $\boldsymbol{SP^{\theta}}$: For each $P \in \mathcal{P}^N$, $SP^{\theta}(P) \equiv \pi \in \Delta(F)$ defined as follows: $\pi_{\theta(1)}(\tau(P_{\theta(1)}, O)) \equiv 1$, $\pi_{\theta(2)}(\tau(P_{\theta(2)}, O \backslash \tau(P_{\theta(1)}, O))) \equiv 1$, and so on.

The next solution is defined by uniformly randomizing over all $n!$ possible elements of Θ and then applying the relevant sequential priority solution.

Random priority solution, $\boldsymbol{RP}$: For each $P \in \mathcal{P}^N$, $RP(P) \equiv \frac{1}{n!} \sum_{\theta \in \Theta} SP^{\theta}(P)$.

Proposition 6. The solutions WF^{sd}, for each $\theta \in \Theta$, SP^{θ}, and RP are *implementable in Nash equilibria.*

Proof. We prove that each of these solutions is *invariant to support monotonic transformation.* $\boldsymbol{WF^{sd}}$: Let $P \in \mathcal{P}^N$ and $\pi \in WF^{sd}(P)$. Then, there is no pair $i, j \in N$ such that $\pi_j \; P_i^{sd} \; \pi_i$. That is, for each pair $i, j \in N$, either (i) $\pi_i = \pi_j$ or (ii) there is $o \in O$ such that $\pi_i(o) > 0$ and $\pi_i(L(P_i, o)) < \pi_j(L(P_i, o))$. Let $P' \in \mathcal{P}^N$ be a *support monotonic transformation of P at π.* Then, for each $k \in N$ and each $t \in O$ such that $\pi_k(t) > 0$, $L(P_k, t) \subseteq L(P'_k, t)$. In case (ii), $\pi_j(L(P'_i, o)) \geq \pi_j(L(P_i, o)) > \pi_i(L(P_i, o)) = \pi_i(L(P'_i, o))$. The last equality comes from the fact that P' is a *support monotonic transformation of p at π.* Thus, for each pair $i, j \in N$, either (i) $\pi_i = \pi_j$ or (ii) there is $o \in O$ such that $\pi_i(o) > 0$ and $\pi_i(L(P'_i, o)) < \pi_j(L(P'_i, o))$, and so $\pi \in WF^{sd}(P')$.

$\boldsymbol{(SP^{\theta})_{\theta \in \Theta}}$ and $\boldsymbol{RP}$: it is straightforward so we omit the proof. □

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