Efficient and Strategy-proof Social Choice When Preferences Are Single-dipped*

Vikram Manjunath†

First draft: February 2008
This version: May 19, 2013

Abstract

We study the problem of locating a single public good along a segment when agents have single-dipped preferences. We ask whether there are *unanimous* and *strategy-proof* rules for this model. The answer is positive and we characterize all such rules.

We generalize our model to allow the set of alternatives to be unbounded. If the set of alternatives does not have a maximal and a minimal element, there is no meaningful notion of *efficiency*. However, we show that the range of every *strategy-proof* rule has a maximal and a minimal element. We then characterize all *strategy-proof* rules.

JEL classification: D71

Keywords: single-dipped preferences, strategy-proofness, group strategy-proofness.

1 Introduction

An agent's preference over the location of a public good along a single dimension is *single-dipped* when he has a single worst point, his "dip", and his welfare increases as the chosen location moves away from this point in either direction.

*I thank William Thomson, Dolors Berga, Bettina Klaus, participants at the SED 2008 Conference on Economic Design, the associate editor and two anonymous referees for their helpful comments and suggestions.

†Département de sciences économiques, Université de Montréal

A few instances of single-dipped preferences are where alternatives are combinations of two goods that have negative cross-effects and the location of a "public bad."

Pareto-efficiency and *strategy-proofness* have been studied in the context of single-dipped preferences over a private good (e.g Klaus, Peters and Storcken (1997), Klaus (2001)). They have been found to result in dictatorship.

For public goods, when the set of alternatives is two dimensional, Öztürk, Peters and Storcken (2012a) and Öztürk, Peters and Storcken (2012b) have shown that, again, *Pareto-efficiency* and *strategy-proofness* generally lead dictatorship.

Since preferences are single-dipped, when the set of alternatives is one dimensional, only the two extremes may be top ranked by an agent. The two axioms that we are interested in imply that the range of a rule be restricted to only the two extremes. Thus, the problem is reduced to choosing between exactly two distinct alternatives. Consequently, our characterization closely resembles the characterization of *voting by extended committees*[1] based on *Pareto-efficiency* and *strategy-proofness* for social choice over two alternatives by Larsson and Svensson (2006).

Interestingly, any *strategy-proof* rule also satisfies the more demanding requirement that no group of agents may all be better off by colluding to misrepresent their preferences. This requirement is called *group strategy-proofness*.

This equivalence also holds when the set of alternatives is any subset of the real line (Peremans and Storcken 1999). In fact, single-dipped preferences are one of many domains that satisfy a condition guaranteeing the equivalence of *group strategy-proofness* and *strategy-proofness* (Barberà, Berga and Moreno 2010, Le Breton and Zaporozhets 2009).

For problems where the set of alternatives is unbounded, the size of a *strategy-proof* rule's range has previously been shown to be bounded by the cardinality of the powerset of agents (Peremans and Storcken 1999). We sharpen this bound to two. We do so by showing that the range of a *strategy-proof* rule has a maximal and a minimal element, observing that *group strategy-proofness* implies *unanimity* over the range, and finally appealing to our earlier characterization. By doing so, we characterize the class of all *strategy-proof* rules. While this class of rules has already been characterized by Peremans and Storcken (1999), we provide a more precise formulation. Barberà, Berga and Moreno (2012b) have subsequently given an alternative proof of this result and studied conditions on the domain of single-dipped preferences for which it holds. The results by Barberà, Berga and Moreno (2012a) apply to rules with binary ranges: the proof strategy in this paper is to show that the range of a *strategy-proof* and *unanimous* rule is necessarily binary.

[1] This is a class of rules that generalizes *voting by committees* (Barberà, Sonnenschein and Zhou 1991) by allowing indifference between the two alternatives.

A stronger notion of immunity to group deviations, which we call *strong group strategy-proofness*, requires that no group of agents may collude to misreport their preferences in such a way that makes *at least one* of them better off without hurting any of the rest. We characterize all *unanimous* and *strongly group strategy-proof* rules.

The rest of the paper is organized as follows. We present the model in Section 2, formal definitions of the relevant axioms in Section 3, and several examples of rules in Section 4. In Section 5 we prove our main results. In Section 6 we generalize our results by dropping *unanimity* and generalize our model by allowing an unbounded set of alternatives.

2 The Model

The set of alternatives is the interval $[0, T]$ in the real line $\mathbb{R}$. The set of agents is N. Given a preference relation R_0 over $\mathbb{R}$, we denote *strict* preference by P_0 and *indifference* by I_0. We say R_0 is **single-dipped** if there exists $d_0 \in \mathbb{R}$ such that for each pair $x, y \in \mathbb{R}$, if $x > y \geq d_0$ or $x < y \leq d_0$, then $x \; P_0 \; y$. Let $\mathcal{R}$ be the set of all single-dipped preferences.[2] Note that these preferences may be discontinuous: upper and lower contour sets need not be closed. Our results hold regardless of whether we allow discontinuous preferences.

A *profile of preferences* associates each agent with a preference relation in $\mathcal{R}$. $\mathcal{R}^N$ is the set of all preference profiles. Given a profile $R \in \mathcal{R}^N$, for each agent $i \in N$, we denote i's preference relation by R_i. We use the notation R_{-i} to denote the preference relations of all agents but i. For each group $S \subseteq N$, we denote the preferences of all the agents in S by R_S, and those not in S by R_{-S}. We denote the set of all sub-profiles of preferences for agents in the group S by $\mathcal{R}^S$.

It is useful to note that for a single-dipped preference relation, the median is never the best among three distinct points.

A **rule** is a function $\varphi : \mathcal{R}^N \to [0, T]$.

Remark 1. *A simple restriction of $\mathcal{R}$ is the set, $\mathcal{R}^{dist} \subset \mathcal{R}$, of preference relations that depend only on the distance between an alternative and the worst point: that is, $R_0 \in \mathcal{R}^{dist}$ if for each pair $x, y \in \mathbb{R}$, $x \; I_0 \; y$ if and only if $|d_0 - x| = |d_0 - y|$.*

For any $x \in [0, T]$, there is a preference relation in $\mathcal{R}^{dist}$ such that its lower contour set at 0 is $[0, x]$. There is also a preference relation that has, as its lower

[2]Single-dippedness of preferences can be generalized to more than one dimension in many ways. If the set of alternatives that can be top ranked under a permissible preference relation has more than two alternatives and its members can be ranked, relative to each other, in any way, we obtain the dictatorial results of (Gibbard 1973) and (Satterthwaite 1975). If we consider single-dipped preferences over a circle, they are also single-peaked, but lead to dictatorial results as well (Schummer and Vohra 2002).

contour set at T the interval $[x, T]$. This is all that is needed for the proofs in Section 5 to hold. Consequently, our results hold for any domain that contains $\mathcal{R}^{dist}$.

Before we state our axioms, we define some notation. For each $R \in \mathcal{R}^N$, let

$$\begin{aligned} N_0(R) &\equiv \{i \in N : 0 \; P_i \; T\}, \\ N_T(R) &\equiv \{i \in N : T \; P_i \; 0\}, \\ N_{0T}(R) &\equiv \{i \in N : 0 \; I_i \; T\}, \\ n_0(R) &\equiv |N_0(R)|, \\ n_T(R) &\equiv |N_T(R)|, \text{and} \\ n_{0T}(R) &\equiv |N_{0T}(R)|. \end{aligned}$$

For each $R_0 \in \mathcal{R}$ and $x \in [0, T]$, let $L(R_0, x)$ be the points that are no better than x at preference relation R_0. That is, $L(R_0, x) \equiv \{y \in [0, T] : x \; R_0 \; y\}$. As a consequence of single-dippedness, for each $R_0 \in \mathcal{R}$ and each $x \in [0, T]$, $L(R_0, x)$ is an interval with one extreme at x. While it is closed at x, since R_0 is possibly discontinuous, it may not be closed at the other extreme.

Let $\overline{L}(R_0, x)$ be the closure of $L(R_0, x)$. The following definitions are depicted in Figure 1. For each $R \in \mathcal{R}^N$ such that $N_0(R) \neq \varnothing$, let

$$\begin{aligned} \overline{x}(R) &\equiv \max_{i \in N_0(R)} \{\min(\overline{L}(R_i, T)\} \text{ and} \\ \overline{I}(R) &\equiv \operatorname*{argmax}_{i \in N_0(R)} \{\min(\overline{L}(R_i, T)\}. \end{aligned}$$

Note that $\underline{I}$ is set-valued. Let

$$\overline{i}(R) = \begin{cases} i \in \overline{I}(R) \text{ such that } \overline{x}(R) \notin L(R_i, T) & \text{if such an } i \text{ exists.} \\ i \in \overline{I}(R) \text{ such that for each } j \in \overline{I}(R), i \leq j & \text{otherwise} \end{cases}$$

For each $R \in \mathcal{R}^N$ such that $N_T(R) \neq \varnothing$, lt

$$\begin{aligned} \underline{x}(R) &\equiv \min_{i \in N_T(R)} \{\max(\overline{L}(R_i, 0)\} \text{ and} \\ \underline{I}(R) &\equiv \operatorname*{argmin}_{i \in N_T(R)} \{\max(\overline{L}(R_i, 0)\}. \end{aligned}$$

Again, $\overline{I}$ is set-valued. Let

$$\underline{i}(R) = \begin{cases} i \in \underline{I}(R) \text{ such that } \underline{x}(R) \notin L(R_i, 0) & \text{if such an } i \text{ exists.} \\ i \in \underline{I}(R) \text{ such that for each } j \in \underline{I}(R), i \leq j & \text{otherwise} \end{cases}$$

For $R \in \mathcal{R}^N$ such that $N_0(R) = \varnothing$ let $\overline{x}(R) = 0$. For $R \in \mathcal{R}^N$ such that $N_T(R) = \varnothing$ let $\underline{x}(R) = T$.

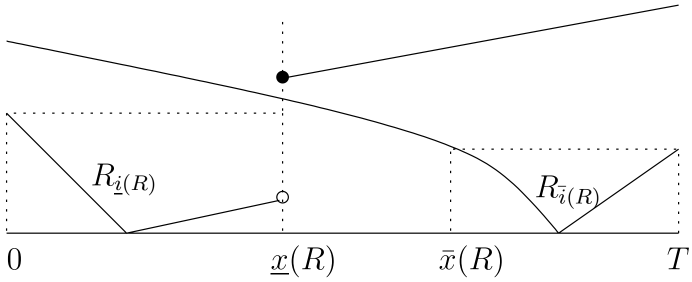

Figure 1: A profile of preferences where only $R_{\underline{i}(R)}$ and $R_{\bar{i}(R)}$ are shown, along with $\underline{x}(R)$ and $\overline{x}(R)$. Note that $L(R_{\underline{i}(R)}, 0)$ is open.

Given a $N' \subseteq N$, $R \in \mathcal{R}^N$, and a pair $x, y \in \mathbb{R}$, let $x\ R_{N'}\ y$ if and only if each member of N' finds x to be as good as y at the preference profile R. That is, for each $i \in N'$, $x\ R_i\ y$. We define $x\ P_{N'}\ y$ and $x\ I_{N'}\ y$ similarly.

For each $R_0 \in \mathcal{R}$ and $A \subset [0, T]$, let $\tau(A, R_0)$ be the elements of A that are most preferred at preference relation R_0. That is,

$$\tau(A, R_0) = \{x \in A : \text{for each } y \in A, x\ R_0\ y\}.$$

3 Axioms

We define some relevant axioms below with respect to a rule φ.

Our first axiom expresses a very mild form of efficiency. It requires a rule to select an alternative that is top-ranked by every single agent, whenever such an alternative exists.

Unanimity: For each $R \in \mathcal{R}^N$, if $N_{0T}(R) \neq N$ and there is $e \in \{0, T\}$ such that $N = N_e(R)\ \cup\ N_{0T}(R)$, then $\varphi(R) = e$.

The usual, stronger, version of efficiency says that a point should not be chosen by the rule if there is another point that makes at least one agent better off without making any of the rest worse off.

For each pair $x, y \in [0, T]$ and each $R \in \mathcal{R}^N$, ***x* Pareto-dominates *y* at *R*** if $x\ R_N\ y$ and for at least one $i \in N$, $x\ P_i\ y$. We denote this as $x\ \bar{P}\ y$.

For each $R \in \mathcal{R}^N$, an alternative $x \in [0, T]$ is **Pareto-efficient at *R*** if there is no $y \in [0, T]$ that *Pareto-dominates* x at R.

For each $R \in \mathcal{R}^N$, let the set of *Pareto-efficient* alternatives at R be $P(R)$. In the appendix, we describe the structure of $P(R)$ in more detail.

Pareto-efficiency: For each $R \in \mathcal{R}^N$, $\varphi(R) \in P(R)$.

The next axiom expresses the requirement that strategic behavior by an agent

never pays: He can do no better than reporting his preferences truthfully.
Strategy-proofness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$,

$$\varphi(R)\ R_i\ \varphi(R'_i, R_{-i}).$$

The following two axioms state that strategic behavior by *groups* of agents does not pay.
Group strategy-proofness: For each $R \in \mathcal{R}^N$, each $S \subseteq N$, and each $R'_S \in \mathcal{R}^S$, there is $i \in S$ such that

$$\varphi(R)\ R_i\ \varphi(R'_S, R_{-S}).$$

Strong group strategy-proofness: For each $R \in \mathcal{R}^N$, each $S \subseteq N$, each $R'_S \in \mathcal{R}^S$, and each $i \in S$,

$$\varphi(R)\ R_i\ \varphi(R'_S, R_{-S}).$$

Our final axiom says that no agent can affect the selection of the rule, by misreporting his preferences, without affecting his own welfare. That is, no agent can be "bossy."
Non-bossiness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$,

$$\varphi(R)\ I_i\ \varphi(R'_i, R_{-i}) \Rightarrow \varphi(R) = \varphi(R'_i, R_{-i}).$$

It turns out that *strong group strategy-proofness* is equivalent to the combination of *non-bossiness* and *strategy-proofness*. It is obvious that *strong group strategy-proofness* implies *strategy-proofness*. First we show that it also implies *non-bossiness*. Then, we show that the *strategy-proofness* and *non-bossiness* together imply *strong group strategy-proofness*.

Lemma 1. *Strong group strategy-proofness implies non-bossiness.*

Proof: Let φ be a *strongly group strategy-proof* rule. Suppose φ is *bossy*. Then there is $i \in N$ and $R \in \mathcal{R}^N$ for which there is $R'_i \in \mathcal{R}$ such that,

$$y \equiv \varphi(R'_i, R_{-i})\ I_i\ \varphi(R) \equiv x.$$

By *strong group strategy-proofness*, there is $j \in N \setminus \{i\}$ such that $y\ P_j\ x$. Thus $\{i, j\}$ are a group such that $y\ R_i\ x$, and $y\ P_j\ x$. This violates *strong group strategy-proofness*. $\square$.

Lemma 2. *Non-bossiness and strategy-proofness together imply strong group strategy-proofness.*

Proof: Let φ be a *non-bossy* and *strategy-proof* rule. Suppose, φ is not *strongly group strategy-proof*. Then, there is $S \subseteq N$ and $R \in \mathcal{R}^N$ for which there is $R_S \in \mathcal{R}^S$ such that for each $i \in S$, $\varphi(R'_S, R_{-S})\ R_i\ \varphi(R)$ and there is $i \in S$ such that $\varphi(R'_S, R_{-S})\ P_i\ \varphi(R)$. Let $x \equiv \varphi(R)$ and $y \equiv \varphi(R'_S, R_{-S})$. Partition S into $S_1 \equiv \{i \in S : y\ I_i\ x\}$ and $S_2 \equiv \{i \in S : y\ P_i\ x\}$. By *group strategy-proofness*, S_1 is nonempty. Let $S_1 \equiv \{1, 2, \ldots, s_1\}$. By *strategy-proofness*, $\varphi(R'_1, R_1) = \varphi(R) = x$. Similarly, $\varphi(R'_1, R'_2, R_{-\{1,2\}}) = x$ and so on. Thus, $\varphi(R'_{S_1}, R_{-S_1}) = x$. But for each $i \in S_2$, $y = \varphi(R'_{S_2}, R'_{S_1}, R_{-S})\ P_i\ \varphi(R'_{S_1}, R_{-S_1}) = x$. This violates *group strategy-proofness*. □

By Lemmas 1 and 2 , *strong group strategy-proofness* is equivalent to the combination of *non-bossiness* and *strategy-proofness*.

In Section 4 we will see that there are rules that are *strategy-proof* but not *strongly group strategy-proof*.

4 Rules

Let $\hat{\mathcal{R}} \subseteq \mathcal{R}$ be such that for each $\hat{R}_0 \in \hat{\mathcal{R}}$, $0\ \hat{I}_0\ T$. Define a **tie-breaker** as any function $t : \hat{\mathcal{R}}^N \to \{0, T\}$.

We start with a simple example of the main class of rules that we later characterize.

Definition: Define the **Majority voting rule, *M***, by setting, for each $R \in \mathcal{R}^N$,

$$M(R) = \begin{cases} 0 & \text{if } n_0(R) \geq n_T(R) \\ T & \text{if } n_0(R) < n_T(R) \end{cases}$$

○

A *majority voting* rule is *Pareto-efficient* and *strategy-proof*.

The essence of *majority voting* is that any group of at least half of the agents can guarantee that 0 (or T) is chosen. We call such groups **0-decisive** (or **T-decisive**). We can generalize this idea by specifying families of groups that are *0-decisive*. For each $M \subset N$, a **family of 0-decisive sets over *M***, $\mathcal{F}_M$, is a family of subsets of M satisfying:

1. **Monotonicity:** For each $S \in \mathcal{F}_M$ and $S' \subseteq M$ such that $S \subseteq S'$, we have $S' \in \mathcal{F}_M$, and

2. **Non-emptiness:** if $M \neq \varnothing$, then $\mathcal{F}_M \neq \varnothing$ and $\varnothing \notin \mathcal{F}_M$. If $M = \varnothing$, then $\mathcal{F}_M = \varnothing$.

A **collection of 0-decisive sets**, $\mathcal{F} \equiv \{\mathcal{F}_M\}_{M\subseteq N}$, is a collection of *families of 0-decisive sets* indexed by subsets of N satisfying the following properties for each $M \subseteq N$ and each $i \in M$:

1. If $S \cup \{i\} \notin \mathcal{F}_M$, then $S \notin \mathcal{F}_{M\setminus\{i\}}$. That is, if $S \cup \{i\}$ is not decisive at M, then S is not decisive at $M \setminus \{i\}$.
2. If $S \in \mathcal{F}_M$ and $i \notin S$, then $S \in \mathcal{F}_{M\setminus\{i\}}$. That is, if S is decisive at M and does not contain i, then S is decisive at $M \setminus \{i\}$.

Definition: Given a collection $\mathcal{F}$ and a tie-breaker t, define **voting by 0-decisive collection $\mathcal{F}$ and tie-breaker t, $VZD^{\mathcal{F},t}$**, by setting, for each $R \in \mathcal{R}^N$,

$$VZD^{\mathcal{F},t}(R) = \begin{cases} t(R) & \text{if } N_{0T}(R) = N \\ 0 & \text{if } N_0(R) \in \mathcal{F}_{N\setminus N_{0T}(R)} \\ T & \text{otherwise.} \end{cases}$$

○

Each member of the class of *voting by 0-decisive collections* rules is *Pareto-efficient* and *strategy-proof* (see Lemma 11).

Remark 2. *A potential objection to the class of voting by 0-decisive collections rules is that they treat the two extremes* 0 *and* T *differently. That is, they exclude rules defined as follows. Let* $\mathcal{F}$ *be a collection of 0-decisive sets and* t *be a tie-breaker. For each* $R \in \mathcal{R}^N$*, let*

$$\varphi^{\mathcal{F},t}(R) = \begin{cases} t(R) & \textit{if } N_{0T}(R) = N \\ T & \textit{if } N_T(R) \in \mathcal{F}_{N\setminus N_{0T}(R)} \\ 0 & \textit{otherwise.} \end{cases}$$

To address this objection, we show that there is a mapping ψ *from the set of all collections of 0-decisive sets to itself, such that for each collection* $\mathcal{F}$*, tie-breaker* t *and* $R \in \mathcal{R}^N$*,*

$$\varphi^{\mathcal{F},t}(R) = VZD^{\psi(\mathcal{F}),t}(R).$$

Before we actually define ψ*, for each* $M \subseteq N$ *and family of 0-decisive sets over* M*,* $\mathcal{F}_M$*, the set of* ***minimal winning groups in*** $\boldsymbol{\mathcal{F}_M}$ *is the set groups in* $\mathcal{F}_M$ *such that no sub-group is also in* $\mathcal{F}_M$*. That is,*

$$MW(\mathcal{F}_M) \equiv \{S \in \mathcal{F}_M : \textit{for each } T \subset S, T \notin \mathcal{F}_M\}.$$

Define ψ *as follows. For each collection* $\mathcal{F}$ *and* $M \subseteq N$*, let*

$$\psi(\mathcal{F})_M = \{T \subseteq M : \textit{for each } S \in MW(\mathcal{F}_M), T \cap S \neq \varnothing\}.$$

First, we check that for each $M \subseteq N$, $\psi(\mathcal{F})_M$ is a family of 0-decisive sets over M. The monotonicity property holds by definition. Since $\cup_{S \in MW(\mathcal{F}_M)} S \in \psi(\mathcal{F}_M)$, $\psi(\mathcal{F}_M)$ is non-empty.

Second, we check that $\psi(\mathcal{F})$ is a collection of 0-decisive sets. Let $M \subseteq N$ and $i \in M$ and $i \in M$.

1. *Suppose $T \cup \{i\} \notin \psi(\mathcal{F})_M$. By definition of ψ, there is $S \in MW(\mathcal{F}_M)$ such that $(T \cup \{i\}) \cap S = \varnothing$. Since $\mathcal{F}$ is a collection of 0-decisive sets, $S \in \mathcal{F}_{M \setminus \{i\}}$. Since $T \cap S = \varnothing$, by definition of ψ, $T \notin \psi(\mathcal{F})_{M \setminus \{i\}}$.*

2. *Suppose $T \in \psi(\mathcal{F})_M$ and $i \notin T$. Let $S \in MW(\mathcal{F}_{M \setminus \{i\}})$. Since $\mathcal{F}$ is a collection of 0-decisive sets, $S \cup \{i\} \in \mathcal{F}_M$. By definition of the set of minimal winning groups, there is $S' \in MW(\mathcal{F}_M)$ such that $S' \subseteq S \cup \{i\}$. Since $T \in \psi(\mathcal{F})_M$, $T \cap S' \neq \varnothing$. Thus, $T \cap S \neq \varnothing$. So $T \in \psi(\mathcal{F})_M$.*

Now we show that for each $R \in \mathcal{R}^N$,

$$\varphi^{\mathcal{F},t}(R) = VZD^{\psi(\mathcal{F}),t}(R).$$

If $N_{0T}(R) = N$, then by definition, $\varphi^{\mathcal{F},t}(R) = t = VZD^{\psi(\mathcal{F}),t}(R)$. Suppose $N \setminus N_{0T}(R) = M \neq \varnothing$. If $\varphi^{\mathcal{F},t}(R) = T$, then $N_T(R) \in \mathcal{F}_M$. Thus there is $S_{min} \in MW(\mathcal{F}_M)$ such that $S_{min} \subseteq N_T(R)$. By definition of ψ, for each $S_\psi \in \psi(\mathcal{F})_M$ and $S_{min} \in MW(\mathcal{F}_M)$, we have $S_\psi \cap S_{min} \neq \varnothing$. Thus, $S_\psi \cap N_T(R) \neq \varnothing$ and $N_0(R) \notin \psi(\mathcal{F})_M$. This means that $VZD^{\psi(\mathcal{F}),t}(R) = T$.

If $\varphi^{\mathcal{F},t}(R) = 0$, then $N_T(R) \notin \mathcal{F}_M$ and there is no $S \in MW(\mathcal{F}_M)$ such that $S \subseteq N_T(R)$. That is, for each $S \in MW(\mathcal{F}_M)$, there is $i_S \in N$ such that $i_S \in S \setminus N_T(R)$. So, $\{i_S\}_{S \in MW(\mathcal{F}_M)} \subseteq N_0(R)$ and for each $S \in MW(\mathcal{F}_M)$, $N_0(R) \cap S \neq \varnothing$. By definition of ψ, $N_0(R) \in \psi(\mathcal{F}_M)$ and $VZD^{\psi(\mathcal{F}),t}(R) = 0$.

Notice that the rules in the class of *voting by 0-decisive collections* depend on the identities of the agents who are indifferent between 0 and T, but not on their entire preference relations unless every agent is indifferent. This is because the *collection of 0-decisive sets* do not depend on these preferences. We can broaden the class by letting the *collection of 0-decisive sets* be a function of the entire preference relation of each agent who is indifferent between 0 and T.

Define an **extended collection of 0-decisive sets** as a function $\mathcal{G}$ mapping profiles of preferences to *collections of 0-decisive sets* such that for each $R \in \mathcal{R}^N$, $\mathcal{G}(R)$ is a *family of 0-decisive sets* over $N \setminus N_{0T}(R)$ and satisfies the following properties:

1. For each $R' \in \mathcal{R}^N$ such that $N_{0T}(R) = N_{0T}(R')$ and $R_{N_{0T}(R)} = R'_{N_{0T}(R)}$, we have $\mathcal{G}(R) = \mathcal{G}(R')$. That is, $\mathcal{G}$ depends on the preferences of agents who are indifferent between 0 and T and their preferences, but does not depend on the preferences of the remaining agents.

2. If $i \notin S$ and $S \cup \{i\} \notin \mathcal{G}(R)$, then for each $R'_i \in \mathcal{R}$ such that $0 \; I'_i \; T$, $S \notin \mathcal{G}(R'_i, R_{-i})$. That is, if $S \cup \{i\}$ is not a decisive set at a preference profile where i is not indifferent between 0 and T, then S is not a decisive set when i's preference is changed in a way that makes him indifferent between the two points and all other agents' preferences are left unchanged.

3. If $S \in \mathcal{G}(R)$ and $i \notin S \cup N_{0T}(R)$, then for each $R'_i \in \mathcal{R}$ such that $0 \; I'_i \; T$, $S \in \mathcal{G}(R'_i, R_{-i})$. That is, if S does not include i and is a decisive set at a preference profile where he is not indifferent between 0 and T, then it is decisive when i's preference is changed in a way that makes him indifferent between the two points and all other agents' preferences are left unchanged.

Definition: For a given *extended collection of 0-decisive sets* $\mathcal{G}$ and *tie-breaker* t, define **voting by extended collection of 0-decisive sets $\boldsymbol{\mathcal{G}}$ with tie-breaker $\boldsymbol{t}$, $\boldsymbol{VEZD^{\mathcal{G},t}}$**, by setting, for each $R \in \mathcal{R}^N$,

$$VEZD^{\mathcal{G},t}(R) = \begin{cases} t(R) & \text{if } N_{0T}(R) = N \\ 0 & \text{if } N_0(R) \in \mathcal{G}(R) \\ T & \text{otherwise.} \end{cases}$$

○

We will show that for each *extended collection of 0-decisive sets* $\mathcal{G}$ and *tie-breaker* t, $VEZD^{\mathcal{G},t}$ is *Pareto-efficient* and *strategy-proof* (Lemma 11).

This is a very broad class of rules. For instance, *majority voting* rules are included in among *VEZD* rules. In fact, we will show that all *unanimous* and *strategy-proof* rules are members of this class (see Lemma 10).

Remark 3. *This class contains rules that depend on the preference relations of agents who are indifferent between 0 and T. Such rules may be meaningful in certain contexts. For instance, in an environment where there is imprecision in the effect of the rule's choice, it makes sense to be sensitive to cases where agents who are indifferent between the* 0 *and T prefer points that are near one extreme to points near the other.*

It is easy to see that not every *VEZD* rule is *strongly group strategy proof.*[3] The following subclass *is strongly group strategy-proof.*

Definition: Given a default point $d \in \{0, T\}$, define the **consensus rule with default $\boldsymbol{d}$, $\boldsymbol{C^d}$**, by setting, for each $R \in \mathcal{R}^N$,

$$C^d(R) = \begin{cases} 0 & \text{if } N_0(R) \neq \varnothing \text{ and } N_T(R) = \varnothing \\ T & \text{if } N_0(R) = \varnothing \text{ and } N_T(R) \neq \varnothing \\ d & \text{otherwise.} \end{cases}$$

[3]Consider a *majority voting* rule. Let $N \equiv \{1, 2, 3\}$ and $R \in \mathcal{R}^N$ be such that $N_0(R) = \{1\}, N_{0T}(R) = \{2\}$, and $N_T(R) = \{3\}$. Let $R'_2 \in \mathcal{R}$ be such that $T \; P'_2 \; 0$. Then $T = M(R'_2, R_{-2}) \; P_3 \; M(R) = 0$. Thus, M is not *strongly group strategy-proof.*

$\circ$

These rules correspond to two extreme *collections of 0-decisive sets*: 1. for each $M \subseteq N$, only M is decisive, and 2. for $M \subseteq N$, each subset of M is decisive. At the end of Section 5 we show these are the only rules that are both *unanimous* and *strongly group strategy-proof* (Corollary 1).

5 Results

An intuitive reason why *unanimity* and *strategy-proofness* do not result in dictatorship is that the most preferred outcome under each preference relation in $\mathcal{R}$ is either 0 or T. This allows us to escape from common negative results since *unanimity* does not imply *onto-ness*, but only that 0 and T are in the range.

The following theorem is the main result. We state it here and subsequently prove it via a series of Lemmas. In particular, it follows immediately from Lemmas 10 and 11. At the end of this section, we consider the implication of strengthening *strategy-proofness* to *strong group strategy-proofness*.

Theorem 1. *φ is* unanimous *and* strategy-proof *if and only if there are an extended collection of 0-decisive sets $\mathcal{G}$ and a tie-breaker t such that φ is* extended voting by collection of 0-decisive sets $\mathcal{G}$ with tie-breaker t.

The axioms of *unanimity* and *strategy-proofness* are independent. A rule with a singleton range is *strategy-proof* but not *unanimous*. The following rule $\hat{\varphi}$ is *unanimous* but not *strategy-proof*: For each $R \in \mathcal{R}^N$ let

$$\hat{\varphi}(R) = \begin{cases} \frac{T}{2} & \text{if } N_0(R) \neq \varnothing \text{ and } N_T(R) \neq \varnothing, \\ 0 & \text{if } N_T(R) = \varnothing, \text{ and} \\ T & \text{otherwise.} \end{cases}$$

We now turn to a series of lemmas that constitute a proof of Theorem 1

Lemma 3. ***(Median never-best):*** *For each $R_0 \in \mathcal{R}$ and $\{x, y, z\} \subset [0, T]$, $med(x, y, z) \notin \tau(\{x, y, z\}, R_0)$.*[4]

Proof: Let $\{x, y, z\} \subset [0, T]$ such that $x < y < z$. If $d_0 \geq y$ then $x \; P_0 \; y$. Otherwise $z \; P_0 \; y$. Thus, $y \notin \tau(A, R_0)$. □

[4] In fact, any preference relation over $[0, T]$ such that the median is never the best among three distinct points is single-dipped. To see this, note that if the median is never best, then, at every point, the lower contour set is convex, and indifference is between no more than two distinct points. These are sufficient conditions for single-dippedness.

Let φ be a *strategy-proof* rule. The following lemma states the equivalence of *strategy-proofness* and *group strategy-proofness*.[5]

Lemma 4. ***(Group strategy-proofness):*** *φ is* group strategy-proof.

Let φ be a rule satisfying *unanimity* in addition to *strategy-proofness*. Lemmas 5 through 10 are with regards to φ.

Lemma 5. ***(Limited range restriction):*** *For each $R \in \mathcal{R}^N$,*

$$\varphi(R) \notin (0, \underline{x}(R)) \quad \textit{and}$$
$$\varphi(R) \notin (\overline{x}(R), T).$$

Proof: If either $N_0(R) = \varnothing$ or $N_T(R) = \varnothing$ then by *unanimity* $\varphi(R) \notin (0, T)$.

Suppose that neither $N_0(R) = \varnothing$ nor $N_T(R) = \varnothing$ and that $\varphi(R) \equiv x \in (0, \underline{x}(R))$. Let $\tilde{R}_{N_T(R)} \in \mathcal{R}^{N_T(R)}$ be such that, as shown in Figure 2,

$$N_T(\tilde{R}_{N_T(R)}, R_{-N_T(R)}) = \varnothing.$$

By *unanimity*, $\varphi(\tilde{R}_{N_T(R)}, R_{-N_T(R)}) \in \{0, T\}$. By the definition of $\underline{x}(R)$, for each $i \in N_T(R)$, $0 \; P_i \; x$ and $T \; P_i \; x$. Thus, for each $i \in N_T(R), \varphi(\tilde{R}_{N_T(R)}, R_{-N_T(R)}) \; P_i \; \varphi(R) = x$. This contradicts *group strategy-proofness*.

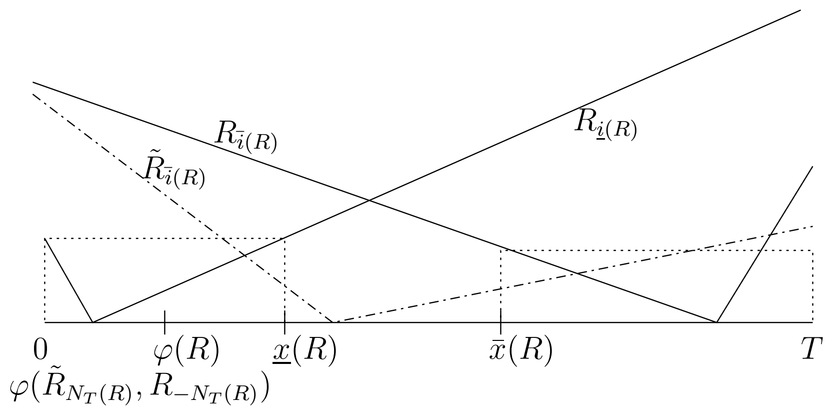

Figure 2: Lemma 5. Limited range restriction: If $\varphi(R) \in (0, \underline{x}(R))$ then $N_T(R)$ can benefit by reporting preferences $\tilde{R}_{N_T(R)}$ such that 0 is chosen instead.

An analogous proof shows that $\varphi(R) \notin (\overline{x}(R), T)$. □

[5] This is equivalent to Lemma 1 of Peremans and Storcken (1999) so we omit the proof. It can also be obtained as an application of the results of Barberà et al. (2010) and Le Breton and Zaporozhets (2009).

Lemma 6. ***(Invariance when $\underline{x} > \overline{x}$ for fixed N_0, N_T, and $R_{N_{0T}}$):*** *For each pair $R, R' \in \mathcal{R}^N$ such that $N_0(R) = N_0(R') = N_0$, $N_T(R) = N_T(R') = N_T$, $R_{N_{0T}(R)} = R'_{N_{0T}(R)}$, $\underline{x}(R) > \overline{x}(R)$, and $\underline{x}(R') > \overline{x}(R')$,*

$$\varphi(R) = \varphi(R').$$

Proof: If $N_0(R) = N_0(R') = \varnothing$ but $N_T(R) = N_T(R') \neq \varnothing$, then by *unanimity* $\varphi(R) = T = \varphi(R')$. Similarly, if $N_0(R) = N_0(R') \neq \varnothing$ but $N_T(R) = N_T(R') = \varnothing$, then by *unanimity* $\varphi(R) = 0 = \varphi(R')$. If $N_0(R) = N_0(R') = \varnothing = N_T(R) = N_T(R')$, then since $N_{0T}(R) = N_{0T}(R'), R = R'$. Thus, $\varphi(R) = \varphi(R')$.

Otherwise, since $\underline{x}(R) > \overline{x}(R)$ and $\underline{x}(R') > \overline{x}(R')$. At least one of the following is true:

$$\overline{x}(R') < \underline{x}(R) \quad (1)$$
$$\underline{x}(R') > \overline{x}(R) \quad (2)$$

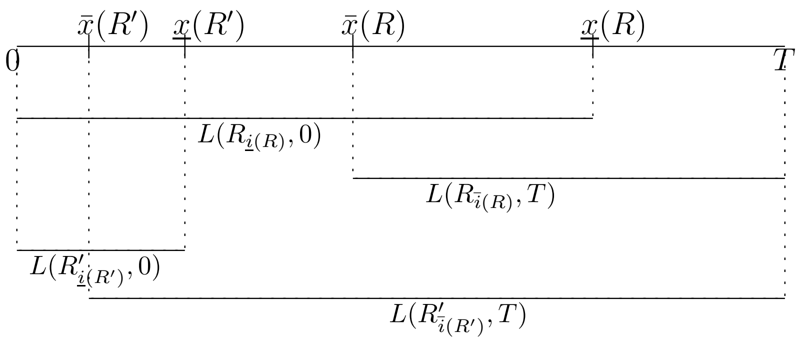

Figure 3: Lemma 6. Invariance when $\underline{x} > \overline{x}$ for fixed N_0, N_T, and $R_{N_{0T}}$: Since $\underline{x}(R) > \overline{x}(R)$ and $\underline{x}(R') > \overline{x}(R')$, we have $\underline{x}(R) > \overline{x}(R')$ and $\underline{x}(R') < \overline{x}(R)$.

Suppose (1) is true, as in Figure 3. By *limited range restriction*, $\varphi(R)$ and $\varphi(R_{N_T}, R'_{N_0}, R_{N_{0T}})$ are both in $\{0, T\}$. Suppose $\varphi(R_{N_T}, R'_{N_0}, R_{N_{0T}}) \neq \varphi(R)$. If $\varphi(R) = T$, then every member of N_0 is better off when it reports R'_{N_0} at the profile R. If $\varphi(R) = 0$, then every member of N_0 is better off when it reports R_{N_0} at the profile $(R_{N_T}, R'_{N_0}, R_{N_{0T}})$. This contradicts *group strategy-proofness*. So $\varphi(R_{N_T}, R'_{N_0}, R_{N_{0T}}) = \varphi(R)$. Similarly, $\varphi(R') = \varphi(R_{N_T}, R'_{N_0}, R_{N_{0T}})$. Thus, $\varphi(R) = \varphi(R')$.

If (1) is false, then (2) is true. In this case, the above argument can be repeated, by switching the roles of R and R'. □

Lemma 7. ***(Range restriction):*** $(0, T) \cap$ *range*(φ) *is empty.*

Proof: Suppose $x \in (0,T) \cap \text{range}(\varphi)$. Then there is $R \in \mathcal{R}^N$ such that $\varphi(R) = x$. By *unanimity*, $N_0(R) \neq \varnothing$ and $N_T(R) \neq \varnothing$.

Let $\tilde{R} \in \mathcal{R}^N$ such that $N_0(\tilde{R}) = N_0(R) = N_0$, $N_T(\tilde{R}) = N_T(R) = N_T$, $\tilde{R}_{N_{0T}(R)} = R_{N_{0T}(R)}$, and $\underline{x}(\tilde{R}) > \overline{x}(\tilde{R})$. By *limited range restriction*, $\varphi(\tilde{R}) \in \{0, T\}$. Suppose $\varphi(\tilde{R}) = 0$.

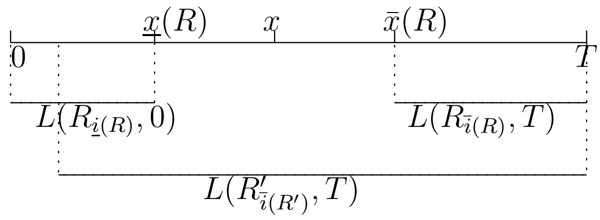

Figure 4: Lemma 7. Range restriction: Find a profile R' such that $\underline{x}(R'_{N_0}, R_{-N_0}) > \overline{x}(R'_{N_0}, R_{-N_0})$.

Let $R'_{N_0} \in \mathcal{R}^{N_0}$ such that for each $i \in N_0$, $T\ P'_i\ \underline{x}(R)$. Then, $\underline{x}(R'_{N_0}, R_{-N_0}) > \overline{x}(R'_{N_0}, R_{-N_0})$, $N_0(R'_{N_0}, R_{-N_0}) = N_0$, and $N_T(R'_{N_0}, R_{-N_0}) = N_T$ as shown in Figure 4. By Lemma 6, $\varphi(R'_{N_0}, R_{-N_0}) = \varphi(\tilde{R})$. But since for each $i \in N_0$, $\varphi(R'_{N_0}, R_{-N_0}) = 0\ P_i\ x = \varphi(R)$ this contradicts *group strategy-proofness*.

A symmetric argument can be used if $\varphi(\tilde{R}) = T$. □

The following is a direct consequence of *unanimity*, *range restriction*, and *strategy-proofness*.

Lemma 8. ***(Invariance for fixed*** N_0, N_T, ***and*** $R_{N_{0T}}$***):*** *For each* $R, R' \in \mathcal{R}^N$ *such that* $N_0(R) = N_0(R')$, $N_T(R) = N_T(R')$, *and* $R_{N_{0T}(R)} = R'_{N_{0T}(R')}$,

$$\varphi(R) = \varphi(R').$$

Lemma 9. ***(Monotonicity):*** *Let* $R, R' \in \mathcal{R}^N$.

If $\varphi(R) = 0$, $N_0(R) \subseteq N_0(R')$, $N_T(R) \supseteq N_T(R')$, *and* $R_{N_{0T}(R) \cap N_{0T}(R')} = R'_{N_{0T}(R) \cap N_{0T}(R')}$, *then* $\varphi(R') = 0$.

Similarly, if $\varphi(R) = T$, $N_0(R) \supseteq N_0(R')$, $N_T(R) \subseteq N_T(R')$, *and* $R_{N_{0T}(R) \cap N_{0T}(R')} = R'_{N_{0T}(R) \cap N_{0T}(R')}$, *then* $\varphi(R') = T$.

Proof: We prove the first part of the lemma. The second part is analogous.

Step 1: Let $R, R^a \in \mathcal{R}^N$ be such that $\varphi(R) = 0$, $N_0(R) \subseteq N_0(R^a)$, $N_T(R) = N_T(R^a)$, and $R^a_{N_{0T}(R^a)} = R_{N_{0T}(R^a)}$. We will show that $\varphi(R^a) = 0$.

Let $S = N_{0T}(R) \setminus N_{0T}(R^a)$. Since $S \subset N_0(R^a)$, by *group strategy-proofness*, $\varphi(R^a_S, R_{-S}) = 0$. By *invariance for fixed* N_0, N_T, *and* $R_{N_{0T}}$, we have $\varphi(R^a) = 0$.

Step 2: Let $R, R^b \in \mathcal{R}^N$ be such that $\varphi(R) = 0, N_0(R) = N_0(R^b), N_T(R) \supseteq N_T(R^b)$, and $R^b_{N_{0T}(R)} = R_{N_{0T}(R)}$. We will show that $\varphi(R^b) = 0$.

Let $S = N_{OT}(R^b) \setminus N_{OT}(R)$. If $\varphi(R^b_S, R_{-S}) \neq 0$, then by *range restriction*, $\varphi(R^b_S, R_{-S}) = T$. This contradicts *group strategy-proofness* since $S \subseteq N_T(R)$ and $\varphi(R) = 0$. Thus $\varphi(R^b_S, R_{-S}) = 0$. By *invariance for fixed* N_0, N_T, *and* $R_{N_{0T}}$, we have $\varphi(R^b) = 0$.

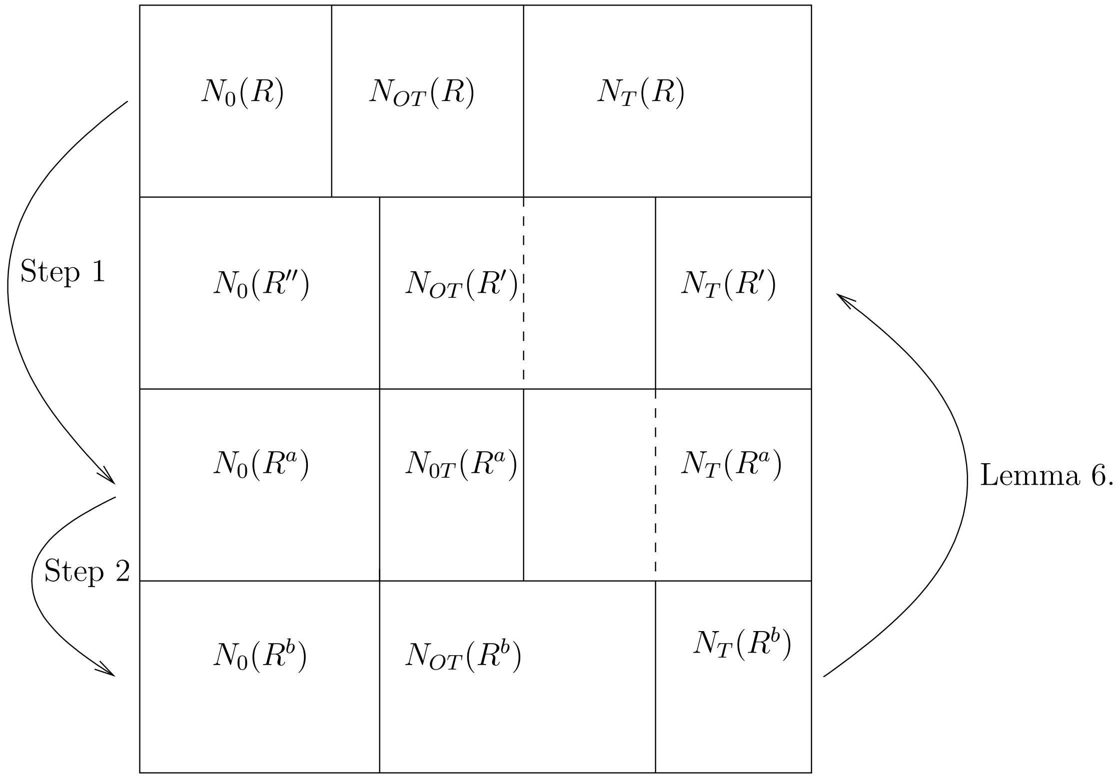

Figure 5: Lemma 9. Monotonicity:Schematic representation of the relationship between N_0, N_T, and N_{0T} at each profile R, R', R^a, and R^b as in step 3.

Step 3: Let $R, R' \in \mathcal{R}^N$ satisfy the hypotheses in the first part of the lemma. We will use steps 1 and 2 to show that $\varphi(R') = 0$ (the procedure is shown in Figure 5). Let $R^a \in \mathcal{R}^N$ be such that $R^a_{N_0(R')} = R'_{N_0(R')}$ and $R^a_{-N_0(R')} = R_{-N_0(R')}$. By Step 1, $\varphi(R^a) = 0$. Let $R^b \in \mathcal{R}^N$ be such that $R^b_{N_T(R')} = R'_{N_T(R')}$ and $R^b_{-N_T(R')} = R^a_{-N_T(R')}$. By Step 2, $\varphi(R^b) = 0$. By *invariance for fixed* N_0, N_T, *and* $R_{N_{0T}}$, we have $\varphi(R') = 0$. □

Lemma 10. ***(VEZD):*** *There is an extended collection of 0-decisive sets* $\mathcal{G}$ *and a tie-breaker* t *such that for each* $R \in \mathcal{R}^N$, $\varphi(R) = VEZD^{\mathcal{G},t}(R)$.

Proof: Define the *extended collection of 0-decisive sets* $\mathcal{G}$ such that for each $R \in \mathcal{R}^N$,

$$\mathcal{G}(R) = \left\{ S \subseteq N \setminus N_{0T}(R) : \begin{array}{l} \text{There is } R'_{-N_{0T}(R)} \in \mathcal{R}^{N \setminus N_{0T}(R)} \text{ such that} \\ N_0(R'_{-N_{0T}(R)}, R_{N_{0T}(R)}) = S, \\ N_T(R'_{-N_{0T}(R)}, R_{N_{0T}(R)}) = N \setminus (N_{0T}(R) \cup S), \text{ and} \\ \varphi(R'_{-N_{0T}(R)}, R_{N_{0T}(R)}) = 0. \end{array} \right\}.$$

Let $t : \hat{\mathcal{R}}^N \to \{0, T\}$ be such that for each $R \in \hat{\mathcal{R}}^N$, $t(R) = \varphi(R)$. By *range restriction*, t is well defined.

Step 1: We show that $\mathcal{G}$ is an extended collection of 0-decisive sets. First we verify that for each $R \in \mathcal{R}^N$, $\mathcal{G}(R)$ is a *family of 0-decisive sets over* $N \setminus N_{0T}(R)$. By *monotonicity*, $\mathcal{G}(R)$ satisfies the monotonicity requirement of a *family of 0-decisive sets*. By *unanimity*, it satisfies the non-emptiness requirement.

We now check the three requirements for $\mathcal{G}$ to be an *extended collection of 0-decisive sets*. For each $R \in \mathcal{R}^N$,

1. By definition of $\mathcal{G}(R)$, it is a function of $R_{N_{0T}(R)}$ but not $R_{N_0(R)}$ and $R_{N_T(R)}$.
2. If $S \in \mathcal{G}(R)$ and $i \in N_{0T}(R)$, then by *monotonicity* , for each $R'_i \in \mathcal{R}$ such that $0 \; P'_i \; T$ or $T \; P'_i \; 0$, $S \cup \{i\} \in \mathcal{G}(R'_i, R_{-i})$.
3. If $S \in \mathcal{G}(R)$ and $i \notin S \cup N_{0T}(R)$, then by *monotonicity*, for each $R'_i \in \mathcal{R}$ such that $0 \; I'_i \; T$, $S \in \mathcal{G}(R'_i, R_{-i})$.

Step 2: We check that for each $R \in \mathcal{R}^N$, $\varphi(R) = VEZD^{\mathcal{G},t}(R)$. Whenever $N_{0T}(R) = N$, $\varphi(R) = t(R) = VEZD^{\mathcal{G},t}(R)$. Suppose $N_{0T}(R) \neq N$ and $VEZD^{\mathcal{G},t}(R) = 0$. By definition of $VEZD$, $N_0(R) \in \mathcal{G}(R)$. Then, by definition of $\mathcal{G}$, there is $\tilde{R} \in \mathcal{R}^{N \setminus N_{0T}(R)}$ such that $N_0(\tilde{R}, R_{N_{0T}}) = N_0(R)$, $N_T(\tilde{R}, R_{N_{0T}}) = N \setminus (N_{0T}(R) \cup N_0(R)) = N_T(R)$, and $\varphi(\tilde{R}, R_{N_{0T}}) = 0$. By *invariance for fixed* $N_0, N_T, and R_{N_{0T}}$, $\varphi(R) = 0$.

A similar argument shows that if $VEZD^{\mathcal{G},t}(R) = T$, then $\varphi(R) = T$. □

Before stating the main theorem, we verify that all rules in the class of *VEZD* are *Pareto-efficient* and *strategy-proof.*

Lemma 11. *For each extended collection of 0-decisive sets $\mathcal{G}$ and tie-breaker t, $VEZD^{\mathcal{G},t}$ is Pareto-efficient and strategy-proof.*

Proof: *Pareto-efficiency* of $VEZD^{\mathcal{G},t}$ follows directly from the non-emptiness requirement in the definition of a family of 0-decisive sets. Since $VEZD^{\mathcal{G},t}(R) = t(R)$ only when $N_{0T}(R) = N$, this is true for any t.

Suppose $VEZD^{\mathcal{G},t}$ is not *strategy-proof*. Then there are $i \in N$, and $R, R' \in N$ such that $VEZD^{\mathcal{G},t}(R'_i, R_{-i})\ P_i\ VEZD^{\mathcal{G},t}(R)$. Then, $N_{0T}(R) \neq N$.

Case $\boldsymbol{VEZD^{\mathcal{G},t}(R) = 0}$: Then $N_0(R) \in \mathcal{G}(R)$, $T\ P_i\ 0$, and $VEZD^{\mathcal{G},t}(R'_i, R_{-i}) = T$. If $0\ I'_i\ T$, then $N_0(R) = N_0(R'_i, R_{-i}) \in \mathcal{G}(R'_i, R_{-i})$ and thus, $VEZD^{\mathcal{G},t}(R'_i, R_{-i}) = 0$. If either $0\ P'_i\ T$ or $T\ P'_i\ 0$, then $N_{0T}(R'_i, R_{-i}) = N_{0T}(R)$ and $\mathcal{G}(R) = \mathcal{G}(R'_i, R_{-i})$. If $0\ P'_i\ T$, by monotonicity of $\mathcal{G}(R)$, $N_0(R'_i, R_{-i}) = N_0(R) \cup \{i\} \in \mathcal{G}(R'_i, R_{-i})$. If $T\ P'_i\ 0$, then $N_0(R'_i, R_{-i}) = N_0(R) \in \mathcal{G}(R'_i, R_{-i})$. In either case, $VEZD^{\mathcal{G},t}(R'_i, R_{-i}) = 0$.

Case $\boldsymbol{VEZD^{\mathcal{G},t}(R) = T}$: Then $N_0(R) \notin \mathcal{G}(R)$, $0\ P_i\ T$, and $VEZD^{\mathcal{G},t}(R'_i, R_{-i}) = 0$. If $0\ P'_i\ T$ or $T\ P'_i\ 0$, we derive a contradiction as in the previous case. If $0\ I_i\ T$ and $N_0(R'_i, R_{-i}) \in \mathcal{G}(R'_i, R_{-i})$, then $N_0(R'_i, R_{-i}) \in \mathcal{G}(R)$ implying that $VEZD^{\mathcal{G},t}(R) = 0$. □

As stated earlier, Theorem 1 follows from Lemmas 10 and 11.

We now consider the implication of strengthening the requirement of *strategy-proofness* to *strong group strategy-proofness*.

Let φ be a *unanimous* and *strong group strategy-proof* rule. By Theorem 1, there are an *extended collection of 0-decisive sets* $\mathcal{G}$ and a tie-breaker t such that $\varphi \equiv VEZD^{\mathcal{G},t}$.

Lemma 12. *If φ is a unanimous and strongly group strategy-proof rule, then there are a* collection of 0-decisive sets *$\mathcal{F}$ and tie-breaker t such that $\varphi \equiv VZD^{\mathcal{F},t}$.*

Proof: By the main theorem of this paper, there is an *extended collection of 0-decisive sets* $\mathcal{G}$ and a tie-breaker t such that $\varphi \equiv VEZD^{\mathcal{G},t}$. We show that for each $R, R' \in \mathcal{R}^N$ if $N_{0T}(R) = N_{0T}(R')$, then $\mathcal{G}(R) = \mathcal{G}(R')$.

Suppose not. Then there are $R, R' \in \mathcal{R}^N$ such that $N_{0T}(R) = N_{0T}(R')$ and $\mathcal{G}(R) \neq \mathcal{G}(R')$. Without loss of generality, there is $S \in \mathcal{G}(R)$ such that $S \notin \mathcal{G}(R')$. Let $\tilde{R} \in \mathcal{R}^N$ be such that $N_0(\tilde{R}) = S$, $N_{0T}(\tilde{R}) = N_{0T}(R)$, and $\tilde{R}_{N_{0T}(R)} = R_{N_{0T}(R)}$. By definition,

$$\begin{aligned} VEZD^{\mathcal{G},t}(\tilde{R}_{-N_{0T}(R)}, R_{N_{0T}(R)}) &= 0 \text{ and} \\ VEZD^{\mathcal{G},t}(\tilde{R}_{-N_{0T}(R)}, R'_{N_{0T}(R)}) &= T. \end{aligned}$$

This contradicts *strong group strategy-proofness*. □

By *range restriction* and *invariance with respect to $R_{N_{0T}}$*, the problem is reduced to the selection of a public good from two alternatives. For this environment only *consensus* rules are *onto* (which is implied by *unanimity*) and *strongly group strategy-proof* (Manjunath 2012). We restate the result here.

Corollary 1. *A rule φ satisfies* unanimity *and* strong group strategy-proofness *if and only if there are a default $d \in \{0, T\}$ and a tie-breaker t such that φ coincides with the* consensus rule with default d and tie-breaker t *whenever at least one agent is not indifferent between 0 and T.*

6 More general sets of alternatives

Suppose that the set of alternatives is now $\mathbb{R}$. A rule is a function $\varphi : \mathcal{R}^N \to \mathbb{R}$.

The definition of *strategy-proofness*, *group strategy-proofness*, *strong group strategy-proofness*, and *non-bossiness* are analogous to the definitions for the more restrictive model. However, none of the *efficiency* notions are well defined. First consider *unanimity*. For each $R_0 \in \mathcal{R}^N$, there is no top ranked alternative in $\mathbb{R}$ under R_0. Thus, *unanimity* has no bite. Next, consider *Pareto-efficiency*. Fix $R_0 \in \mathcal{R}$. Let $R \in \mathcal{R}^N$ be such that for each $i \in N$, $R_i = R_0$. The only *Pareto-efficient* allocation at R is the top-ranked alternative in $\mathbb{R}$ under R_0. Since there is no such alternative, there is no *Pareto-efficient* allocation at R.

We next show that the range of a *strategy-proof* rule has a maximal and a minimal element. Let φ be a *strategy-proof* rule. The proof of the *group strategy-proofness* lemma follows through unchanged. Thus, φ is *group strategy-proof.*

We now show that the range has a maximal and a minimal element.[6]

Lemma 13. ***(Range limits)*** *There are $\underline{x}^\varphi$, $\overline{x}^\varphi \in$ range(φ) such that for each $x \in$ range(φ),*

$$\underline{x}^\varphi \leq x \leq \overline{x}^\varphi.$$

Proof: Let $R^0 \in \mathcal{R}^N$ be such that for each $i \in N$, $d_i = 0$ and for each $x \in \mathbb{R}$, $x \; I_i \; -x$. Let $x^0 \equiv \varphi(R^0)$.
Case $\boldsymbol{x^0 > 0}$: Suppose there is $x \in \text{range}(\varphi)$ such that $x > x^0$. Then there is $R \in \mathcal{R}^N$ such that $x = \varphi(R)$. Since $x > x^0$, for each $i \in N, x = \varphi(R) \; P_i^0 \; \varphi(R^0) = x^0$. This violates *group strategy-proofness* of φ. Set $\overline{x}^\varphi \equiv x^0$. Thus, there is no $x \in \text{range}(\varphi)$ such that $x > \overline{x}^\varphi$.

Let $\tilde{R} \in \mathcal{R}^N$ be such that for each $i \in N$, $\tilde{d}_i > \overline{x}^\varphi$. Let $\tilde{x} \equiv \varphi(\tilde{R})$. By the previous step, $\tilde{x} \leq \overline{x}^\varphi$. Suppose there is $R \in \mathcal{R}^N$ such that $\varphi(R) < \tilde{x}$. Then for each $i \in N$, $\varphi(R) \; \tilde{P}_i \; \varphi(\tilde{R}) = \tilde{x}$. This violates *group strategy-proofness* of φ. Set $\underline{x}^\varphi \equiv \tilde{x}$. Thus, there is no $x \in \text{range}(\varphi)$ such that $x < \underline{x}^\varphi$.
Case $\boldsymbol{x^0 < 0}$: The proof is analogous to the previous case.
Case $\boldsymbol{x^0 = 0}$: Here, we show that the range$(\varphi) = \{x^0\}$. If there is $R \in \mathcal{R}^N$ such that $\varphi(R) \neq x^0$, then for each $i \in N$, $\varphi(R) \; P_i^0 \; x^0$. This violates *group strategy-proofness* of φ. □

Let us normalize so that $\underline{x}^\varphi = 0$ and $\overline{x}^\varphi = T$.

Since φ is *group strategy-proof*, $\{0, T\} \subseteq \text{range}(\varphi) \subseteq [0, T]$, φ is a *unanimous* rule for problems with the range $[0, T]$. Since it is also *strategy-proof*, we can appeal to Theorem 1 and conclude that there is an extended collection of 0-decisive sets

[6] This is implied by Lemma 2 of Peremans and Storcken (1999), but we provide a simple direct proof.

$\mathcal{G}$ and a tie-breaker t such that φ is extended voting by collection of 0-decisive sets $\mathcal{G}$ with tie-breaker t.

In other words, we can characterize the following class of rules.

Let $\underline{a}, \overline{a} \in \mathbb{R}$ be such that $\underline{a} \leq \overline{a}$. Let $\mathcal{G}$ be an *extended collection of $\underline{a}$-decisive sets* and $t : \mathcal{R}^N \to \{\underline{a}, \overline{a}\}$ be a tie-breaker. Define **voting between $\underline{a}$ and $\overline{a}$ by extended collection of $\underline{a}$-decisive sets $\mathcal{G}$ with tie-breaker t, $V\underline{A}\overline{A}ED^{\underline{a},\overline{a},\mathcal{G},t}$**, by setting, for each $R \in \mathcal{R}^N$,

$$V\underline{A}\overline{A}ED^{\underline{a},\overline{a},\mathcal{G},t}(R) = \begin{cases} t(R) & \text{if for each } i \in N, \underline{a}\ I_i\ \overline{a} \\ \underline{a} & \text{if } \{i \in N : \underline{a}\ P_i\ \overline{a}\} \in \mathcal{G}(R) \\ \overline{a} & \text{otherwise.} \end{cases}$$

Corollary 2. *φ is* strategy-proof *if and only if there are $\underline{a}, \overline{a} \in \mathbb{R}$,* extended collection of $\underline{a}$-decisive sets $\mathcal{G}$*, and tie-breaker t such that φ is* voting between $\underline{a}$ and $\overline{a}$ by extended collection of $\underline{a}$-decisive sets $\mathcal{G}$ with tie-breaker t.

Appendices

A The Pareto-set

The following lemmas pertain to the nature of the *Pareto set*.

Lemma 14. ***(Domination by extremes):*** *For each $R \in \mathcal{R}^N$ and each $x \notin P(R)$, either $0\ \bar{P}\ x$ or $T\ \bar{P}\ x$.*

Proof: Let $R \in \mathcal{R}^N$. If $x \notin P(R)$, then there is $y \in [0, T]$ such that $y\ \bar{P}\ x$. If $x < y$, then $y = \text{med}\{x, y, T\}$. Since $y\ \bar{P}\ x$, $y\ R_N\ x$. By single-dippedness of preferences, $T\ P_N\ y$. Therefore, $T\ \bar{P}\ x$.

If $x > y$, an analogous argument shows that $0\ \bar{P}\ x$. □

Lemma 15. ***(Shape of the Pareto set):*** *For each $R \in \mathcal{R}^N$, $P(R)$ has one of the following configurations:*

1. *$\{0\}$ if $n_0(R) > 0$ and $n_T(R) = 0$.*

2. *$\{T\}$ if $n_0(R) = 0$ and $n_T(R) > 0$.*

3. *$\{0, T\} \cup (\underline{x}(R), \overline{x}(R))$ if $n_0(R) > 0$, $n_T(R) > 0$, $\underline{x}(R) \in L(R_{\underline{i}(R)}, 0)$, $\overline{x}(R) \in L(R_{\overline{i}(R)}, 0)$, and $\underline{x}(R) < \overline{x}(R)$*

4. *$\{0, T\} \cup [\underline{x}(R), \overline{x}(R))$ if $n_0(R) > 0$, $n_T(R) > 0$, $\underline{x}(R) \notin L(R_{\underline{i}(R)}, 0)$, $\overline{x}(R) \in L(R_{\overline{i}(R)}, 0)$, and $\underline{x}(R) < \overline{x}(R)$*

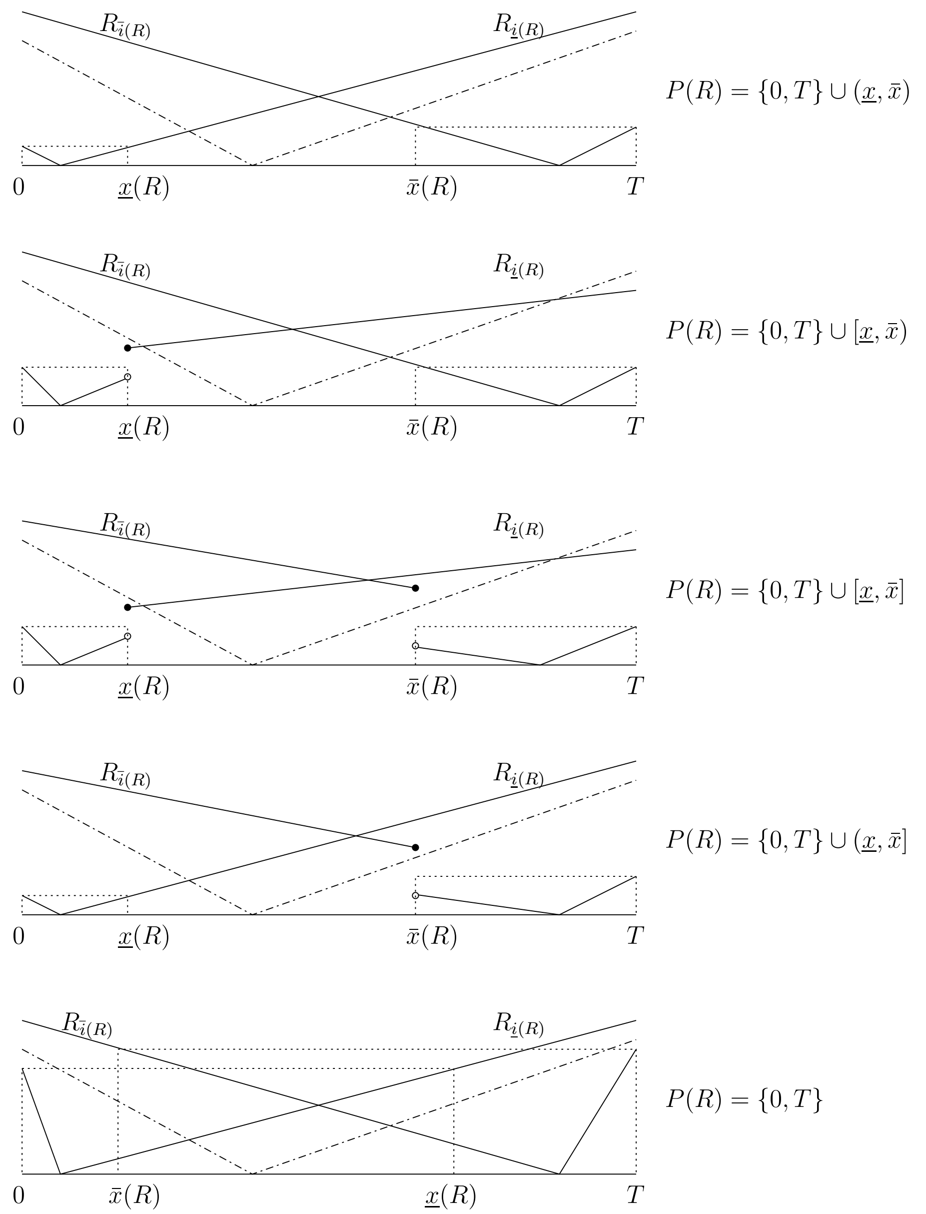

Figure 6: Five profiles of preferences showing the Pareto sets described in cases 3 through 7 of Lemma 15.

5. $\{0, T\} \cup [\underline{x}(R), \overline{x}(R)]$ *if* $n_0(R) > 0$*,* $n_T(R) > 0$*,* $\underline{x}(R) \notin L(R_{\underline{i}(R)}, 0)$*,* $\overline{x}(R) \notin L(R_{\bar{i}(R)}, 0)$*, and* $\underline{x}(R) < \overline{x}(R)$

6. $\{0, T\} \cup (\underline{x}(R), \overline{x}(R)]$ *if* $n_0(R) > 0$*,* $n_T(R) > 0$*,* $\underline{x}(R) \in L(R_{\underline{i}(R)}, 0)$*,* $\overline{x}(R) \notin L(R_{\bar{i}(R)}, 0)$*, and* $\underline{x}(R) < \overline{x}(R)$

7. $\{0,T\}$ *if* $n_0(R) > 0$*,* $n_T(R) > 0$*, and* $\underline{x}(R) > \overline{x}(R)$*, or* $N_{0T} = N$*.*

Proof:
Case 1: $n_0(R) > 0, n_T(R) = 0$. For each $x \in (0,T], 0\ \bar{P}\ x$. Thus, $P(R) = \{0\}$.
Case 2: $n_T(R) > 0, n_0(R) = 0$. For each $x \in [0,T), T\ \bar{P}\ x$. Thus, $P(R) = \{T\}$.
Case 3: $N_{0T} = N$. For each $x \in (0,T)$, $0\ \bar{P}\ x$ and $0\ I_N\ T$. Thus, $P(R) = \{0,T\}$.
Case 4: $n_0(R) > 0, n_T(R) > 0$. By definition of $\underline{i}(R)$ and $\overline{i}(R)$, $d_{\underline{i}(R)} \leq \underline{x}(R)$ and $d_{\overline{i}(R)} \leq \overline{x}(R)$.

Step 1: For each $y \in (0, \underline{x}(R))$, $0\ \bar{P}\ y$. First, $0\ P_{N_0(R)}\ T$. Thus, $0\ P_{N_0(R)}\ y \equiv \text{med}\{0,y,T\}$. Second, $0\ P_{N_{0T}(R)}\ y$. Finally, $0\ R_{N_T(R)}\ y$. Suppose not. Then there is $j \in N_T(R)$ such that $y\ P_j\ 0$. By definition $\underline{x}(R) = \min_{i \in N_T(R)} \{\max(\overline{L}(R_i, 0)\}$. Thus, for each $i \in N_T(R)$, $y \in \overline{L}(R_i, 0)$. Thus for each $i \in N_T(R)$, $0\ P_i\ y$.

Step 2: For each $y \in (\overline{x}(R), T)$, $T\ \bar{P}\ y$. The proof is analogous to that of Step 1.

Step 3: If $\underline{x}(R) < \overline{x}(R)$, then $(\underline{x}(R), \overline{x}(R)) \subset P(R)$. Let $y \in (\underline{x}(R), \overline{x}(R))$. Then $y\ P_{\overline{i}(R)}\ T$ and $y\ P_{\underline{i}(R)}\ 0$. Thus, neither $0\ \bar{P}\ y$ nor $T\ \bar{P}\ y$. By *domination by extremes*, $y \in P(R)$.

Step 4: $\{0,T\} \subseteq P(R)$. For each $i \in N_0(R)$, $\tau([0,T], R_i) = \{0\}$ and, for each $j \in N_T(R), \tau([0,T], R_j) = \{T\}$. Thus, $\{0,T\} \subseteq P(R)$.

Step 5: If $\underline{x}(R) < \overline{x}(R)$, and $\underline{x}(R) \notin L(R_{\underline{i}(R)}, 0)$ then $\underline{x}(R) \in P(R)$. Since $\underline{x}(R) \notin L(R_{\underline{i}(R)}, 0)$, $\underline{x}(R)\ P_{\underline{i}(R)}\ 0$. Since $\underline{x}(R) < \overline{x}(R)$, $\underline{x}(R)\ P_{\overline{i}(R)}\ T$. By *domination by extremes*, $\underline{x}(R) \in P(R)$.

Step 6: If $\underline{x}(R) < \overline{x}(R)$, and $\overline{x}(R) \notin L(R_{\overline{i}(R)}, 0)$ then $\overline{x}(R) \in P(R)$. The proof is analogous to that of Step 5. □

References

Barberà, Salvador, Dolors Berga, and Bernardo Moreno (2010) "Individual versus group strategy-proofness: When do they coincide?" *Journal of Economic Theory*, Vol. 145, No. 5, pp. 1648–1674, September.

Barberà, Salvador, Dolors Berga, and Bernardo Moreno (2012a) "Group strategy-proof social choice functions with binary ranges and arbitrary domains: characterization results," *International Journal of Game Theory*, Vol. 41, pp. 791–808.

——— (2012b) "Domains, ranges and strategy-proofness: the case of single-dipped preferences," *Social Choice and Welfare*, Vol. 39, No. 2, pp. 335–352, July.

Barberà, Salvador, Hugo Sonnenschein, and Lin Zhou (1991) "Voting by Committees," *Econometrica*, Vol. 59, No. 3, pp. 595–609, May.

Gibbard, Allan (1973) "Manipulation of Voting Schemes: A General Result," *Econometrica*, Vol. 41, No. 4, pp. 587–601.

Klaus, Bettina (2001) "Coalitional Strategy-Proofness in Economies with Single-Dipped Preferences and the Assignment of an Indivisible Object," *Games and Economic Behavior*, Vol. 34, pp. 64–82.

Klaus, Bettina, Hans Peters, and Ton Storcken (1997) "Strategy-proof division of a private good when preferences are single-dipped," *Economic Letters*, Vol. 55, pp. 339–346.

Larsson, Bo and Lars-Gunnar Svensson (2006) "Strategy-proof voting on the full preference domain," *Mathematical Social Sciences*, Vol. 52, No. 3, pp. 272–287, Dec.

Le Breton, Michel and Vera Zaporozhets (2009) "On the equivalence of coalitional and individual strategy-proofness properties," *Social Choice and Welfare*, Vol. 33, No. 2, pp. 287–309, Aug.

Manjunath, Vikram (2012) "Group Strategy-proofness And Voting Between Two Alternatives," *Mathematical Social Sciences*, Vol. 63, No. 3, pp. 239–242.

Öztürk, Murat, Hans Peters, and Ton Storcken (2012a) "On the location of public bads: strategy-proofness under two-dimensional single-dipped preferences," working paper, Maastricht : METEOR, Maastricht Research School of Economics of Technology and Organization.

——— (2012b) "Strategy-proof location of a public bad on a disc," working paper, Maastricht : METEOR, Maastricht Research School of Economics of Technology and Organization.

Peremans, Wouter and Ton Storcken (1999) "Strategy-proofness on Single-Dipped Preference Domains," *Proceedings of the International Conference, Logic, Game Theory and Social Choice*, pp. 296–313.

Satterthwaite, Mark (1975) "Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions," *Journal of Economic Theory*, Vol. 10, pp. 187–217.

Schummer, James and Rakesh V Vohra (2002) "Strategy-proof Location on a Network," *Journal of Economic Theory*, Vol. 104, No. 2, pp. 405–428, Jun.