Group Strategy-proofness And Voting Between Two Alternatives[☆]

Vikram Manjunath[1]

Abstract

We study rules for choosing between two alternatives when people may be indifferent between them. We specify two strategic requirements for groups of people. The first, *group strategy-proofness*, says that manipulations by groups ought not make every member of the group better off. The second, *strong group strategy-proofness*, says that such manipulations ought not make at least one member of the group better off without making another worse off. Our main result is a characterization of "consensus" rules and "constant" rules as the only *strongly group strategy-proof* rules when there are more than two people.
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1. Introduction

We study rules for choosing between two alternatives when the people involved may be indifferent between them. In particular, we consider the implications of requiring that a rule cannot be profitably manipulated by groups of people. We specify such a requirement in two different ways: *Group strategy-proofness* says that no group can collude to misreport its preferences in a way that makes every member better off; *Strong group strategy-proofness* says that no group can collude to misreport its preferences in a way that makes at least one member of the group better off without making other members worse off.

If a rule is *strongly group strategy-proof* then it is obviously *group strategy-proof*. Rules that satisfy either of these requirements are clearly *strategy-proof*.

There are many settings where groups make binary choices. Decisions regarding whether or not to convict a defendant, confirm an appointee, or implement a project are only a few examples. With few exceptions (Larsson and Svensson, 2006) most of the literature has assumed away indifference. However, there are real-world situations where

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[1]Département de sciences économiques, Université de Montréal.

people's preferences do exhibit indifference. If preferences are based on coarse descriptions, there may be insufficient information to break ties. Alternatively, if preferences are based on checklists of criteria, distinct alternatives satisfying exactly the same criteria are equivalent. When preferences do not involve indifference, *group strategy-proofness* is, by definition, equivalent to *strong group strategy-proofness*.

A class of rules that generalizes "voting by committees" (Barberà et al., 1991), which we refer to as *voting by x-decisive sets*,[2] consists of all rules that are *strategy-proof* and *onto* (Larsson and Svensson, 2006).

We show that, for this model, every *strategy-proof* rule is *group strategy-proof* (Proposition 1) but not necessarily *strongly group strategy-proof*.[3] Our main result is a characterization of "consensus" rules and "constant" rules as the only *strongly group strategy-proof* rules (Corollary 1). Since *constant* rules are the only ones to violate "onto-ness," we establish our result by showing that there are, in *welfare* terms, only two rules that are *strongly group strategy-proof* and *onto*:

1. Choose the first alternative if *every* person finds it to be at least as desirable as the other. Otherwise, pick the second alternative.
2. Choose the first alternative if *at least one* person finds it to be at least as desirable as the other. Otherwise, pick the second alternative.

This result is specific to settings with exactly two alternatives to choose from. When there are at least three alternatives, only *dictatorial* rules are *strategy-proof* and *onto* (Gibbard, 1973; Satterthwaite, 1975; Beja, 1993). While such rules are *group strategy-proof*, they are not *strongly group strategy-proof*. This establishes that *strong group strategy-proofness* is incompatible with *onto-ness*.

The remainder of the paper is organized as follows. We present the model in Section 2. We formally describe the relevant axioms in Section 3. In Section 4, we describe several classes of rules. We prove our results in Section 5 and conclude in Section 6

2. The Model

Let $\boldsymbol{N}$ be a set of people and $\boldsymbol{O \equiv \{a, b\}}$ be a set of two possible *outcomes*. Each person in N has a preference relation over O: he either prefers a, prefers b, or is indifferent between them. Let $\boldsymbol{\mathcal{R}}$ be the set of these three preference relations.

For each $i \in N$, let $\boldsymbol{R_i} \in \mathcal{R}$ denote i's preference relation. For each pair, $x, y \in O$, if i finds x to be at least as good as y, we write $\boldsymbol{x\ R_i\ y}$. If he prefers x to y, we write $\boldsymbol{x\ P_i\ y}$ and if he is indifferent between the two, $\boldsymbol{x\ I_i\ y}$.

A **problem** is described by a profile $\boldsymbol{R} \in \mathcal{R}^N$.

For each $R \in \mathcal{R}^N$, we use $\boldsymbol{R_{-i}}$ to denote the preference relations of all but i. For each $S \subseteq N$, we denote the preferences of those in S by $\boldsymbol{R_S}$, and those not in S by $\boldsymbol{R_{-S}}$. We denote the set of all sub-profiles of preferences for people in S by $\boldsymbol{\mathcal{R}^S}$.

For each $R \in \mathcal{R}^N$, let $\boldsymbol{N_a(R)}$ be the set of people who prefer a to b at the preference profile R. Similarly, let $\boldsymbol{N_b(R)}$ be the set of people who prefer b to a, and let $\boldsymbol{N_{ab}(R)}$ be the set of people who are indifferent between a and b.

A **rule**, $\boldsymbol{\varphi : \mathcal{R}^N \to O}$, selects an outcome for each profile of preferences.

[2] Larsson and Svensson (2006) call these "voting by extended committees."

[3] For instance, rules like *majority* rules and *serial dictatorship* rules (Section 4) are *group strategy-proof* but not *strongly group strategy-proof*.

3. Axioms

In this section, we list some requirements on rules. Let φ be a rule.

Our first requirement is that each alternative be chosen for some profile of preferences.

Onto-ness: There is a pair $R^a, R^b \in \mathcal{R}$ such that $\varphi(R^a) = a$ and $\varphi(R^b) = b$.

Since there are exactly two alternatives, a rule violates this axiom if and only if it picks the same alternative at every profile. Requiring *onto-ness*, thus, only excludes rules that are maximally unresponsive preferences in this sense.

The next requirement is that no person can benefit by misreporting his preferences.

Strategy-proofness: For each $R \in \mathcal{R}^N$, there is no $i \in N$ for whom there is $R'_i \in \mathcal{R}$ such that $\varphi(R'_i, R_{-i})\ P_i\ \varphi(R_i, R_{-i})$. If such an $i \in N$ and $R'_i \in \mathcal{R}$ exist, we say that **i manipulates φ at R via R'_i.**

Our final requirements are two notions of immunity to collusive misreporting of preferences. First, no group can collude to misreport its preferences in a way that makes every member better off.

Group strategy-proofness: For each $R \in \mathcal{R}^N$, there is no $S \subseteq N$ for which there is $R'_S \in \mathcal{R}^S$ such that, for each $i \in S$, $\varphi(R'_S, R_{-S})\ P_i\ \varphi(R_S, R_{-S})$. If such an $S \subseteq N$ and $R'_S \in \mathcal{R}^S$ exist, we say that **S strongly manipulates φ at R via R'_S.**

The second notion is that no group can collude to misreport its preferences in a way that makes at least one member better off without making other members worse off.

Strong group strategy-proofness: For each $R \in \mathcal{R}^N$, there is no $S \subseteq N$ for which there is $R'_S \in \mathcal{R}^S$ such that, for each $i \in S$, $\varphi(R'_S, R_{-S})\ R_i\ \varphi(R_S, R_{-S})$ and for some $i \in S$, $\varphi(R'_S, R_{-S})\ P_i\ \varphi(R_S, R_{-S})$. If such an $S \subseteq N$ and $R'_S \in \mathcal{R}^S$ exist, we say that **S manipulates φ at R via R'_S .**

4. Rules

For each $x \in O$, define the **constant rule at x, K^x**, by setting, for each $R \in \mathcal{R}^N$, $K^x(R) = x$.

Since they are entirely unresponsive to preferences, *constant rules* are *strongly group strategy-proof*. However, they are the only rules that are not *onto*.

The next class of rules express the idea that one of the two alternatives is favored (perhaps as a "status quo") and the other is chosen only if *every* person agrees. Such rules are common in the real world.[4] For each pair $d_1, d_2 \in O$, define the **consensus rule with disagreement-default d_1 and indifference-default d_2, C^{d_1,d_2}**, by setting, for each $R \in \mathcal{R}^N$,

$$C^{d_1,d_2}(R) = \begin{cases} a & \text{if } N_a(R) \neq \varnothing \text{ and } N_b(R) = \varnothing, \\ b & \text{if } N_a(R) = \varnothing \text{ and } N_b(R) \neq \varnothing, \\ d_1 & \text{if } N_a(R) \neq \varnothing \text{ and } N_b(R) \neq \varnothing, \text{ and} \\ d_2 & \text{if } N_a(R) = \varnothing \text{ and } N_b(R) = \varnothing. \end{cases}$$

Consensus rules are *strongly group strategy-proof* (Lemma 1) and *onto*.

[4] An important example is the requirement of unanimous verdicts for criminal jury trials in many jurisdictions. That is, for such trials, the defendant is not convicted if even a single member of the jury doubts his guilt.

Remark 1. There are *essentially* only two consensus rules: one with *disagreement-default* a and the other with *disagreement-default* b. This is because the choice of *indifference-default*, d_2, has no implications on welfare as it only matters when *every* person is indifferent between a and b. ◇

Let $\prec$ be an ordering over N. Suppose $1 \prec 2 \prec \cdots \prec n$. Further, let $t \in O$. Define the **serial dictatorship rule with respect to ordering $\prec$ and tie-breaker t, $D^{\prec,t}$**, by setting, for each $R \in \mathcal{R}^N$,

$$D^{\prec,t}(R) = \begin{cases} a & \text{if } 1 \in N_a(R), \\ b & \text{if } 1 \in N_b(R), \\ a & \text{if } 1 \in N_{ab}(R) \text{ and } 2 \in N_a(R), \text{ and} \\ b & \text{if } 1 \in N_{ab}(R) \text{ and } 2 \in N_b(R), \\ \vdots & \\ t & \text{if for each } N_{ab}(R) = N. \end{cases}$$

Serial dictatorship rules are *group strategy-proof* and *onto*, but not *strongly group strategy-proof*.[5]

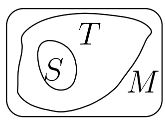
Define the **Majority rule, $\overline{M}$**, by setting, for each $R \in \mathcal{R}^N$,

$$\overline{M}(R) = \begin{cases} a & \text{if } |N_a(R)| \geq |N_b(R)| \text{ and} \\ b & \text{if } |N_a(R)| < |N_b(R)|. \end{cases}$$

The *majority* rule is *group strategy-proof* and *onto*, but not *strongly group strategy-proof*.

A common feature of the *strategy-proof* and *onto* rules above is that there are groups (any group that includes half of $N \setminus N_{ab}(R)$ for $\overline{M}$, $\{1\}$ for $D^{\prec,t}$, N for C^{b,d_2}, and so on) who can guarantee that a is chosen. That is, they can induce the rule to choose a if each of them states a preference for it. We call such groups ***a*-decisive**.[6] We generalize this idea by specifying families of groups that are decisive for one of the two alternatives. For each $M \subset N$, a **family of *a*-decisive sets at M, $\mathcal{F}_M$,** is a family of subsets of M, satisfying the following two properties:

1. **Non-emptiness:** if $M \neq \varnothing$, then $\mathcal{F}_M \neq \varnothing$ and $\varnothing \notin \mathcal{F}_M$. If $M = \varnothing$, then $\mathcal{F}_M = \varnothing$.
2. **Closedness under expansion:** For each $S \in \mathcal{F}_M$ and $T \subseteq M$, if $S \subseteq T$, then $T \in \mathcal{F}_M$, as shown in Figure 1.



$$S \in \mathcal{F}_M \Rightarrow T \in \mathcal{F}_M,$$

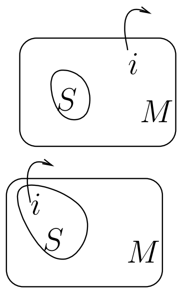
Figure 1: The *closedness under expansion* property in the definition of a *family of a-decisive sets*.

[5] Though, if $|N| = 2$, they *are strongly group strategy-proof*.

[6] We could equivalently work with b-decisive sets.

A **collection of a-decisive sets**, $\mathcal{F} \equiv \{\mathcal{F}_M\}_{M \subseteq N}$, is a collection of *families of a-decisive sets* indexed by subsets of N satisfying the following properties for each $M \subseteq N$ and each $i \in M$, as shown in Figure 2:[7]

1. If $S \in \mathcal{F}_M$ and $i \notin S$, then $S \in \mathcal{F}_{M \setminus \{i\}}$. That is, if S is decisive at M, then S is decisive at $M \setminus \{i\}$.
2. If $S \cup \{i\} \notin \mathcal{F}_M$, then $S \notin \mathcal{F}_{M \setminus \{i\}}$. That is, if $S \cup \{i\}$ is not decisive at M, then S is not decisive at $M \setminus \{i\}$.



$$i \notin S \in \mathcal{F}_M \Rightarrow S \in \mathcal{F}_{M \setminus \{i\}}.$$

$$\{i\} \cup S \notin \mathcal{F}_M \Rightarrow S \notin \mathcal{F}_{M \setminus \{i\}}.$$

Figure 2: The two properties that a *collection of a-decisive sets* satisfies for each $M \subseteq N$ and each $i \in M$.

For each $\mathcal{F}$ and each $t \in O$, define **voting by a-decisive collection $\mathcal{F}$ and tie-breaker t, $VDC^{\mathcal{F},t}$**, by setting, for each $R \in \mathcal{R}^N$,

$$VDC^{\mathcal{F},t}(R) = \begin{cases} t & \text{if } N_{ab}(R) = N \\ a & \text{if } N_a(R) \in \mathcal{F}_{N \setminus N_{ab}(R)} \\ b & \text{otherwise.} \end{cases}$$

Rules in this class are *onto* and *group strategy-proof*. In fact, they are the only ones that are *onto* and *strategy-proof* (Larsson and Svensson, 2006).[8]

Remark 2. It is easy to see that *consensus* rules, *serial dictatorship* rules, and *majority* rules are all instances of *voting by a-decisive collection* rules. For instance, we can define $\mathcal{F}^C$ such that $VDC^{\mathcal{F}^C,b} \equiv C^{a,b}$, by setting, for each $M \subseteq N$ such that $M \neq \varnothing$,

$$\mathcal{F}^C_M \equiv \{M' : M' \subseteq M \text{ and } M' \neq \varnothing\}.$$

Similarly, we can define $\mathcal{F}^{\overline{M}}$ such that $M \equiv VDC^{\mathcal{F}^{\overline{M}},a}$, by setting, for each $M \subseteq N$ such that $M \neq \varnothing$,

$$\mathcal{F}^{\overline{M}}_M \equiv \{M' : M' \subseteq M \text{ and } |M'| \geq |M \setminus M'|\}.$$

◇

[7] For another context where the idea of *collections of decisive sets* plays an important role, see Manjunath (2011).

[8] These rules are equivalent to *voting by extended committees* defined by Larsson and Svensson (2006) (page 278). The only difference between their definition and ours is notation.

5. Results

Our first proposition is the equivalence of *strategy-proofness* and *group strategy-proofness*. This is a corollary of the main results in Le Breton and Zaporozhets (2009) and Barberà et al. (2010). However, we provide a direct proof for this model.

Proposition 1. *Every* strategy-proof *rule is* group strategy-proof*.*

Proof: Let φ be a *strategy-proof* rule. Define a **collusive group** as $S \subseteq N$ for which there are $R \in \mathcal{R}^N$ and $R'_S \in \mathcal{R}^S$ such that S strongly manipulates φ at R via R'_s.

By *strategy-proofness*, no *collusive group* consists of a single person. Suppose there is no *collusive group* of size k. Let $S \subseteq N$ be a *collusive group* of size $k+1$. Without loss of generality, suppose $\varphi(R) = b$. Then, $S \subseteq N_a(R)$. Let $i \in S$. By assumption $S \setminus \{i\}$ is not *collusive*. So $\varphi(R'_{S\setminus\{i\}}, R_i, R_{-S}) = b$. By *strategy-proofness* $\varphi(R'_S, R_{-S}) = b$. This contradicts the definition of S. □

The only *strategy-proof* and *onto* rules are *voting by a-decisive collections* (Larsson and Svensson, 2006). We restate this below.

Theorem 1. *Only VDC-rules are* strategy-proof *and* onto*.*

Remark 3. Recall that the only rules that are not *onto* are *constant* rules. These are trivially *group strategy-proof*. Combining this with Theorem 1, we can indirectly verify the equivalence of *strategy-proofness* and *group strategy-proofness* by checking that each rule in the class of *voting by a-decisive collections* is *group strategy-proof*. ◇

We now show that *consensus* rules are the only *strongly group strategy-proof* and *onto* rules. We begin by showing that they are, in fact, *strongly group strategy-proof*.

Lemma 1. *For each pair $d_1, d_2 \in O$, C^{d_1,d_2} is* strongly group strategy-proof*.*

Proof: Suppose there are $S \subseteq N$, $R \in \mathcal{R}^N$, and $R'_S \in \mathcal{R}^S$ such that S manipulates C^{d_1,d_2} at R via R'_S. Then, $C^{d_1,d_2}(R'_S, R_{-S}) \neq C^{d_1,d_2}(R)$ and $N_{ab}(R) \neq N$.

If $C^{d_1,d_2}(R) = d_1$, then there is $i \in N_{d_1}(R)$ and $i \notin S$. Thus, $N_{d_1}(R'_S, R_{-S}) \neq \varnothing$. So, $C^{d_1,d_2}(R'_S, R_{-S}) = d_1$. This contradicts $C^{d_1,d_2}(R) \neq C^{d_1,d_2}(R'_S, R_{-S})$.

Thus, $C^{d_1,d_2}(R) \neq d_1$. By definition of C^{d_1,d_2}, $N_{d_1}(R) = \varnothing$ and for each $i \in N$, $C^{d_1,d_2}(R)\ R_i\ C^{d_1,d_2}(R'_S, R_{-S})$. This contradicts the assumption that S manipulates C^{d_1,d_2} at R via R'_S. □

The following lemma establishes that a rule is both *strongly group strategy-proof* and *onto* if and only if it is a *consensus* rule. However, this result, and those that follow from it, only holds when there are more than two people. When there are two people, *serial dictatorship* rules also satisfy these properties (Barberà et al., 2011).

Lemma 2. *For $|N| > 2$, if φ is* strongly group strategy-proof *and* onto*, then there is a pair $d_1, d_2 \in O$ such that $\varphi \equiv C^{d_1,d_2}$.*

Proof: Let $\overline{R} \in \mathcal{R}^N$ be such $N_a(\overline{R}) = \{1\}, N_b(\overline{R}) = \{2\}$, and $N_{ab}(\overline{R}) = N \setminus \{1, 2\}$. Let $d_1 \equiv \varphi(\overline{R})$. Without loss of generality, suppose that $d_1 = a$.

Let $\underline{R} \in \mathcal{R}^N$ be such that $N_{ab}(\underline{R}) = N$. Let $d_2 \equiv \varphi(\underline{R})$.

We show that for each $R \in \mathcal{R}^N$, $\varphi(R) = C^{a,d_2}(R)$. There are three cases:

Case 1: $N_a(R) \neq \varnothing$.

We show that $\varphi(R) = a$. Suppose, instead, that $\varphi(R) = b$. There are three sub-cases:

Case 1.1: $1 \in N_a(R)$. For each $i \in N \setminus \{1, 2\}$,

$$b = \varphi(R)\ \overline{I}_i\ \varphi(R_1, \overline{R}_{-1}) = \varphi(\overline{R}) = a, \text{and}$$

$$b = \varphi(R)\ \overline{P}_2\ \varphi(R_1, \overline{R}_{-1}) = \varphi(\overline{R}) = a.$$

This violates *strong group strategy-proofness*: $N \setminus \{1\}$ manipulates φ at $\overline{R}$ via R_{-1}.

Case 1.2: $1 \in N_{ab}(R)$. Let $R'_1 \in \mathcal{R}$ be such that $a\ P'_1\ b$. Let $R' \equiv (R'_1, R_{-1})$. By Case 1.1, $\varphi(R') = a$. However,

$$a = \varphi(R'_1, R_{-1})\ I_1\ \varphi(R) = b,$$

and for each $i \in N_a(R)$,

$$a = \varphi(R'_1, R_{-1})\ P_i\ \varphi(R) = b.$$

This violates *strong group strategy-proofness*: $N_a(R) \cup \{1\}$ manipulates φ at R via $(R'_1, R_{N_a(R)})$.

Case 1.3: $1 \in N_b(R)$. Since $N_a(R) \neq \varnothing$, let $k \in N_a(R)$. Let $R' \in \mathcal{R}^N$ be such that $N_a(R') \equiv \{k\}, N_{ab}(R') = \{1\}$, and $N_b(R') \equiv N \setminus \{1, k\}$. Thus, $R'_k = R_k$. By Case 1.2, $\varphi(R') = a$. However,

$$b = \varphi(R) = \varphi(R_k, R_{-k})\ I'_1\ \varphi(R_k, R'_{-k}) = \varphi(R') = a,$$

and for each $i \in N \setminus \{1, k\}$,

$$b = \varphi(R) = \varphi(R_k, R_{-k})\ P'_i\ \varphi(R_k, R'_{-k}) = \varphi(R') = a.$$

This violates *strong group strategy-proofness*: $N \setminus \{k\}$ manipulates φ at R' via R_{-k}.

Case 2: $N_a(R) = \varnothing$ and $N_b(R) \neq \varnothing$.

We show that $\varphi(R) = b$. Suppose, instead, that $\varphi(R) = a$. Since φ is *onto*, there is $R^b \in \mathcal{R}^N$ such that $\varphi(R^b) = b$. This is violates *strong group strategy-proofness*: N manipulates φ at R via R^b.

Case 3: $N_a(R) = \varnothing$ and $N_b(R) = \varnothing$.

Since $N_{ab}(R) = N$, by definition of d_2, $\varphi(R) = d_2$. □

Combining Lemmas 1 and 2, we have the following theorem.

Theorem 2. *For $|N| > 2$, only* consensus rules *are* strongly group strategy-proof *and* onto*.*

From this theorem, since only *constant* rules are not *onto*, we have the following corollary, which is our main result.

Corollary 1. *For $|N| > 2$, Only* consensus *and* constant *rules are* strongly group strategy-proof*.*

6. Conclusion

The sole axiom imposed in our main result (Corollary 1) is *strong group strategy-proofness*. This axiom is so strong that, when there are more than two people, it rules out all but two constant rules and (essentially) two consensus rules. None of these rules are sensitive to the identities of people. That is, they are "anonymous." It is notable that the formulation of *strong group strategy-proofness* does not (in any obvious way) imply *anonymity*. Yet, as we see from Corollary 1, *anonymity* is implied by it.

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