The Impossibility of Individually Rational, Pareto-efficient, and Strategy-proof rules for Fractional Matching Markets

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Abstract

For a model of fractional two-sided matching, we show that individual rationality, Pareto-efficiency, and strategy-proofness are incompatible. This result is robust to whether, or to what extent, transfers are possible. Since we prove this impossibility for the domain of preferences with linear utility representations, a natural interpretation of the model is of probabilistic matching. We show that even the weaker *ex post* Pareto-efficiency is incompatible with individual rationality and strategy-proofness.

JEL classification: C71, C78, D51

Keywords: *fractional, matching, individual rationality, Pareto-efficiency, strategy-proofness*

1 Introduction

We study a model of fractional matching with two sides (Manjunath, 2016). Each agent has a unit of time available to spend with partners on the other side or by themselves. Each agent is also endowed with a non-negative quantity of a perfectly divisible good that we call money. Preferences are convex, continuous, and increasing in money. We search for rules that are individually rational,[1] Pareto-efficient,[2] and strategy-proof.[3] We find that no such rule exists even if the preferences of each agent is representable as a linear function of the time shares and money (Theorem 1). We prove this impossibility result on the domain of

[1]Individual rationality means that no agent finds remaining single and consuming his or her endowment of money preferable to his or her assignment.

[2]Pareto-efficiency means that no agent can be made better off without making another agent worse off.

[3]Strategy-proofness means that no agent receives a preferable assignment by misreporting his or her preferences.

linearly representable preferences. In fact, we only appeal to such preferences where there are no ties between partners. Thus, our result extends to larger domains that contain it, such as the domain of convex and continuous preferences that are increasing in money.

A necessary condition for a fractional matching to be Pareto-efficient is that it be a convex combination of Pareto-efficient integral matchings. This is the extent to which we rely on Pareto-efficiency to show our impossibility result. Where we interpret fractional matching as probabilistic matching, this means that we may weaken Pareto-efficiency to *ex post* Pareto-efficiency and still conclude with an impossibility (Proposition 1).

For the standard two-sided matching model (Gale and Shapley, 1962), no individually rational and Pareto-efficient rule is strategy-proof (Alcalde and Barberà, 1994). Such a problem can be represented as a fractional matching problem where no agent is endowed with a positive amount of money, feasible allocations are integral, and preferences over partners are strict and have a linear utility representation. Since every *ex post* efficient and deterministic matching is Pareto-efficient, Proposition 1 improves upon the impossibility result of Alcalde and Barberà (1994) by dropping the requirements that the rule be deterministic and only use ordinal preference information.

Our difficulty with obtaining an individually rational, Pareto-efficient, and strategy-proof rule is a familiar one in related settings. Such rules do not exist for exchange economies with classical preferences over at least two divisible private goods (Hurwicz, 1972; Serizawa, 2002). Even weakening the requirement of individual rationality does not permit an escape from the negative result (Zhou, 1991; Schummer, 1997; Ju, 2003; Kato and Ohseto, 2002; Momi, 2017; Cho and Thomson, 2017).[4] The impossibility holds for exchange economies where there are one or more public goods as well (Schummer, 1999).

Since there is a matching component to an allocation, we show an impossibility even when there is only one divisible private good. A fractional matching between two agents bears resemblance to a public good, since its consumption is non-rival. Nevertheless, to rationalize this interpretation, each agent's time would be a private good that has no value to any other agent, except as an input into a Leontieff production technology to produce an excludable public good, the fractional match with another agent.[5] To the best of our knowledge, none of the related results on economies with private and public goods covers such production technology. In exchange economies, Schummer (1997, 1999) and Cho and Thomson (2017) also work with preferences that have linear utility representations. However our underlying models differ and so our results are independent.

[4]Even weakening Pareto-efficiency or strategy-proofness does not help (Schummer, 2004; Cho, 2014).

[5]For Leontieff preferences, on the other hand, positive results have been shown for exchange economies (Nicolò, 2004; Li and Xue, 2013).

For the probabilistic allocation of objects, strategy-proofness is incompatible with Pareto-efficiency and equal treatment of equals (Zhou, 1990).[6] This problem may be encoded in our model by assuming that agents on one side are indifferent between all partners and no agent has a positive endowment of money. However, we show our result for the subdomain where such indifferences are ruled out. Thus, it is independent of the known impossibilities for probabilistic object allocation.

The rest of this paper is organized as follows: we describe the model in Section 2, present and discuss the main impossibility result in Section 3, and apply it to probabilistic matching in Section 4.

2 The Model

A *fractional matching model* consists of a finite set of agents N partitioned into two non-empty sets: M and W.

Each agent has unit availability. For each $m \in M$, a *fractional allocation* divides his availability between partners in W and being alone. That is, a fractional allocation for m is

$$\pi_m \in \Delta_m \equiv \left\{ y_m \in \mathbb{R}_+^{W \cup \{m\}} : \sum_{i \in W \cup \{m\}} y_{mi} = 1 \right\}.$$

For each $w \in W, \pi_{mw}$ represents the time that m spends with w while π_{mm} is the time that he spends alone. For each $w \in W$, define Δ_w similarly and let $\Delta \equiv \underset{i \in N}{\times} \Delta_i$. For each $i \in N$ and each possible partner j for i, let $\delta_i^j \in \Delta_i$ be such that $\delta_{ij}^j = 1$. Analogously define δ_i^i.

Consider a profile of fractional allocations, $\pi \in \Delta$. For each $m \in M$ and $w \in W$, π_{mw} is the time that m spends with w and π_{wm} is the time that w spends with m. For this to make any sense, these must be the same: the time that they spend together. So feasibility requires that $\pi_{mw} = \pi_{wm}$. Such a π is a *fractional matching*. Let Π be the set of all fractional matchings.

Aside from spending time with others, agents also consume a divisible private good that we call money. Each $i \in N$ is endowed with $\omega_i \in \mathbb{R}_+$ units of money. Since i consumes a fractional allocation along with an amount of money, his consumption set is $X_i \equiv \Delta_i \times \mathbb{R}_+$. Let $X \equiv \underset{i \in N}{\times} X_i$. Note that this includes the possibility that for each $i \in N$, $\omega_i = 0$ so that there no possibility of making transfers between agents.

[6] Even if we consider comparisons based on stochastic dominance, corresponding notions of efficiency, fairness, and strategy-proofness are incompatible (Bogomolnaia and Moulin, 2001; Nesterov, 2017) even if the domain of ordinal preferences is restricted(Kasajima, 2013; Chang and Chun, 2016).

An *allocation* is $(\pi, z) \in X$ such that π is a fractional matching and z feasibly allocates all of the money in the problem. That is, $\pi \in \Pi$ and $\sum_{i\in N} z_i = \sum_{i\in N} \omega_i$. Let $\mathcal{Z}$ be the set of allocations.

For each $i \in N$, i's preferences over his consumption set are described by a utility function $u_i : X_i \to \mathbb{R}$. We assume that u_i is continuous and quasi-concave. We also assume that it is monotonic in money. That is, for each $\pi_i \in \Delta_i$ and each pair $z_i, z_i' \in \mathbb{R}_+$, if $z_i > z_i'$ then $u_i(\pi_i, z_i) > u_i(\pi_i, z_i')$. Let $\tilde{\mathcal{U}}_i$ be the set of all such utility functions and let $\tilde{\mathcal{U}} \equiv \times_{i\in N} \tilde{\mathcal{U}}_i$.

We fix M, W, and ω. A *problem* is fully described by a profile of utility functions $u \in \tilde{\mathcal{U}}$. A *domain* of problems, $\mathcal{U}$, is a Cartesian product subset of $\tilde{\mathcal{U}}$. A *rule for* $\mathcal{U}$, $\varphi : \mathcal{U} \to \mathcal{Z}$, is a function that assigns to each problem u in $\mathcal{U}$ a feasible allocation $\varphi(u) \in \mathcal{Z}$.

Domains Given $i \in N$, let $\mathcal{U}_i^{lin}$ be the set of all linear utility functions for i in $\tilde{\mathcal{U}}_i$. In probabilistic matching contexts, these utility functions correspond to von Neumann-Morgenstern preferences. For each $u_i \in \mathcal{U}^{lin}$, let $R(u_i)$ be the preference relation over i's partners induced by u_i. That is, if $i \in M$, let $R(u_i)$ be such that for each pair $j, k \in W \cup \{i\}$, $j \; R(u_i) \; k$ if and only if $u_i(\delta_i^j, 0) \geq u_i(\delta_i^k, 0)$, and similarly if $i \in W$.

Let $\mathcal{U}_i^{slin}$ be the set of all linear utility functions for i that induce *strict* preferences over i's partners. That is, for each $u_i \in \mathcal{U}_i^{slin}$, $R(u_i)$ is a strict preference relation over i's partners.

Then, $\mathcal{U}^{lin} = \times_{i\in N} \mathcal{U}_i^{lin}$ is the domain of problems with linear utility functions and $\mathcal{U}^{slin} = \times_{i\in N} \mathcal{U}_i^{slin}$ is the subdomain of $\mathcal{U}^{lin}$ containing all problems with linear utility *and* strict preferences over partners.

Properties Let $u \in \tilde{\mathcal{U}}$. If $(\pi, z) \in \mathcal{Z}$ is such that for each $i \in N$, $u_i(\pi_i, z_i) \geq u_i(\delta_i^i, \omega_i)$, then (π, z) is *individually rational at* u. If there is no other $(\pi', z') \in \mathcal{Z}$ such that for each $i \in N$, $u_i(\pi_i', z_i') \geq u_i(\pi_i, z_i)$ and for some $i \in N$, $u_i(\pi_i', z_i') > u_i(\pi_i, z_i)$, then (π, z) is *Pareto efficient at* u.

A rule $\varphi : \mathcal{U} \to \mathcal{A}$ is *individually rational* if for each $u \in \mathcal{U}$, $\varphi(u)$ is individually rational at u. It is *Pareto-efficient* if for each $u \in \mathcal{U}$, $\varphi(u)$ is Pareto-efficient at u. It is *strategy-proof* if, for each $u \in \mathcal{U}$ and each $i \in N$, there is no $u_i' \in \mathcal{U}_i$ such that

$$u_i(\varphi_i(u_i', u_{-i})) > u_i(\varphi_i(u)).$$

3 Impossibility result

The following result holds for fractional matching problems with or without money. In the proof, we have only relied upon linear utility functions that induce strict preferences over partners. That the negative result can be shown on such a small domain implies that it is particularly robust: it holds for any domain that contains $\mathcal{U}^{slin}$. The proof is also invariant to the amount of money in the problem.

We illustrate the proof in Figures 1 and 2 for the case without money. When there is money, the need for an additional (fourth) dimension makes a diagrammatic representation infeasible.

Theorem 1 *If $|M| \geq 2$ and $|W| \geq 2$ and $\mathcal{U} \supseteq \mathcal{U}^{slin}$, no strategy-proof rule for $\mathcal{U}$ selects an individually rational and Pareto-efficient and allocation for every problem.*

Proof: We start with the case where $|M| = |W| = 2$. Let $M \equiv \{m_1, m_2\}$ and $W \equiv \{w_1, w_2\}$. Let $\Omega \equiv \sum_{i \in N} \omega_i$, $K > 12\Omega$, and $\alpha > \frac{14K+32\Omega}{K-12\Omega}$. Note that $\alpha > 0$.

Let $u \in \mathcal{U}^{slin}$ be such that for each $(\pi, z) \in \mathcal{Z}$,

$$\begin{aligned}
u_{m_1}(\pi_{m_1}, z_{m_1}) &= K\pi_{m_1w_1} + \tfrac{K}{2}\pi_{m_1w_2} + z_{m_1}, \\
u_{m_2}(\pi_{m_2}, z_{m_2}) &= K\pi_{m_2w_2} + \tfrac{K}{2}\pi_{m_2w_1} + z_{m_2}, \\
u_{w_1}(\pi_{w_1}, z_{w_1}) &= K\pi_{w_1m_2} + \tfrac{K}{2}\pi_{w_1m_1} + z_{w_1}, \text{ and} \\
u_{w_2}(\pi_{w_2}, z_{w_2}) &= K\pi_{w_2m_1} + \tfrac{K}{2}\pi_{w_2m_2} + z_{w_2}.
\end{aligned}$$

Let $(\pi, z) \equiv \varphi(u)$. If there is $m \in M$ such that $\pi_{mm} > 0$ then there is $w \in W$ such that $\pi_{ww} > 0$, by feasibility. So, increasing π_{mw} while decreasing π_{mm} and π_{ww} by the same amount and maintaining feasibility, ceteris paribus, leads to a Pareto-improvement of (π, z). We reach a similar contradiction of the Pareto-efficiency of (π, z) if there is $w \in W$ such that $\pi_{ww} > 0$. Thus, for each $i \in N$, $\pi_{ii} = 0$, so π is a convex combination of fractional matchings σ^1 and σ^2, where $\sigma^1_{m_1w_1} = \sigma^1_{m_2w_2} = 1$ and $\sigma^2_{m_1w_2} = \sigma^2_{m_2w_1} = 1$. Let $l \in [0,1]$ be such that $\pi = l\sigma^1 + (1-l)\sigma^2$.

Suppose that $l \leq \frac{1}{2}$. Let $u'_{m_1} \in \mathcal{U}^{slin}_{m_1}$ be such that for each $(\pi_{m_1}, z_{m_1}) \in X_{m_1}$,

$$u'_{m_1}(\pi_{m_1}, z_{m_1}) = K\pi_{m_1w_1} - \alpha K\pi_{m_1w_2} + z_{m_1}.$$

Let $(\pi', z') \equiv \varphi(u'_{m_1}, u_{-m_1})$. If $\pi'_{m_2m_2} > 0$, then there is $w \in W$ such that $\pi'_{ww} > 0$, by feasibility. So, increasing π'_{m_2w} while decreasing $\pi_{m_2m_2}$ and π_{ww} and maintaining feasibility, ceteris paribus, leads to a Pareto-improvement of (π', z'), a contradiction. So $\pi'_{m_2m_2} = 0$. If $\pi'_{w_1w_1} > 0$, then since $\pi'_{m_2m_2} = 0$, we have that $\pi'_{m_1m_1} > 0$. So, increasing $\pi'_{m_1w_1}$ while decreasing $\pi_{m_1m_1}$ and $\pi_{w_1w_1}$ and maintain feasibility, ceteris paribus, leads to a Pareto-improvement of (π', z'), a contradiction. So $\pi'_{w_1w_1} = 0$. Thus, π' is a convex combination of fractional matchings σ^1, σ^2, and σ^3, where $\sigma^3_{m_1m_1} = \sigma^3_{w_2w_2} = \sigma^3_{m_2w_1} = 1$. This implies that

$$\pi'_{m_1w_1} = \pi'_{m_2w_2}. \tag{1}$$

Notice that $u_{m_1}(\pi_{m_1}, z_{m_1}) = Kl + K(\frac{1-l}{2}) + z_{m_1} = K(\frac{1+l}{2}) + z_{m_1}$. By strategy-proofness,

$$K\pi'_{m_1w_1} + \frac{K}{2}\pi'_{m_1w_2} + z'_{m_1} = u_{m_1}(\pi'_{m_1}, z'_{m_1}) \leq u_{m_1}(\pi_{m_1}, z_{m_1}) = K\left(\frac{1+l}{2}\right) + z_{m_1}.$$

Then

$$K\pi'_{m_1w_1} \leq K\left(\frac{1+l}{2}\right) + z_{m_1} - z'_{m_1} \leq K\left(\frac{1+l}{2}\right) + \Omega,$$

where the final inequality follows from money allocations being non-negative and constrained above by the aggregate endowment. Thus, since $l \leq \frac{1}{2}$ and $K > 12\Omega$,

$$\pi'_{m_1w_1} \leq \frac{3}{4} + \frac{\Omega}{K} < 1. \tag{2}$$

By (1) and (2),

$$u_{m_2}(\pi'_{m_2}, z'_{m_2}) \leq K\left(\frac{3}{4} + \frac{\Omega}{K}\right) + \frac{K}{2}\left(\frac{1}{4} - \frac{\Omega}{K}\right) + \Omega = \frac{7}{8}K + \frac{3}{2}\Omega, \tag{3}$$

where we use the fact that $z_{m_2} \leq \Omega$.

Let $u'_{m_2} \in \mathcal{U}^{slin}_{m_2}$ be such that for each $(\pi_{m_2}, z_{m_2}) \in X_{m_2}$,

$$u'_{m_2}(\pi_{m_2}, z_{m_2}) = K\pi_{m_2w_2} - \alpha K\pi_{m_2w_1} + z_{m_2}.$$

Define $(\pi'', z'') \equiv \varphi(u'_M, u_W)$. If $\pi''_{m_1m_1} > 0$ and $\pi''_{w_1w_1} > 0$, then increasing $\pi_{m_1w_1}$ while decreasing $\pi_{m_1m_1}$ and $\pi_{w_1w_1}$ and maintaining feasibility, ceteris paribus, leads to a Pareto-improvement of (π'', z''), a contradiction. So either $\pi''_{m_1m_1} = 0$ or $\pi''_{w_1w_1} = 0$. Similarly, either $\pi''_{m_2m_2} = 0$ or $\pi''_{w_2w_2} = 0$. Thus, π'' is a convex combination of fractional matchings $\sigma^1, \sigma^2, \sigma^3$, and σ^4, where $\sigma^4_{m_1w_2} = \sigma^4_{m_2m_2} = \sigma^4_{w_1w_1} = 1$. So there are $p, q, r, s \in [0, 1]$ such that $\pi'' = p\sigma^1 + q\sigma^2 + r\sigma^3 + s\sigma^4$.

Since (π'', z'') is individually rational at (u''_M, u_W) and $p = 1 - q + r + s$,

$$u'_{m_1}(\pi''_{m_2}, z''_{m_1}) = Kp - \alpha K(q + s) + z''_{m_1} \geq \omega_{m_1} = u'_{m_1}(\delta^{m_1}_{m_1}, \omega_{m_1}) \tag{4}$$

and

$$u'_{m_2}(\pi''_{m_2}, z''_{m_2}) = Kp - \alpha K(q + r) + z''_{m_2} \geq \omega_{m_2} = u''_{m_2}(\delta^{m_2}_{m_2}, \omega_{m_2}) \tag{5}$$

Adding (4) and (5), we get

$$2Kp - \alpha K(q + r + s) - \alpha Kq + z''_{m_1} + z''_{m_2} \geq \omega_{m_1} + \omega_{m_2},$$

which implies

$$2Kp - \alpha K(1-p) \geq \alpha Kq + \omega_{m_1} + \omega_{m_2} - z''_{m_1} - z''_{m_2},$$

since $p = 1 - q - r - s$. Then, since $\alpha Kq \geq 0$ and $\omega_{m_1} + \omega_{m_2} - z''_{m_1} - z''_{m_2} \geq -\Omega$, we have that $2Kp - \alpha K(1-p) \geq -\Omega$, and so

$$p \geq \frac{\alpha K - \Omega}{(\alpha+2)K}. \tag{6}$$

Note that the right hand side of (6) is less than one since Ω, K, and α are all non-negative. Finally,

$$u_{m_2}(\pi''_{m_2}, z''_{m_2}) = Kp + \frac{K}{2}(q+r) + z''_{m_2} \geq Kp \geq \frac{\alpha K - \Omega}{\alpha+2},$$

where the last inequality follows from (6). However, by definition of α,

$$u_{m_2}(\pi''_{m_2}, z''_{m_2}) \geq \frac{\alpha K - \Omega}{\alpha+2} > \frac{7}{8}K + \frac{3}{2}\Omega \geq u_{m_2}(\pi'_{m_2}, z'_{m_2}),$$

where the last inequality follows from (3). This contradicts strategy-proofness, since m_2 can profitably manipulate with report u'_{m_2} at the profile (u'_{m_1}, u_{-m_1}).

Thus, $l > \frac{1}{2}$. However, we reach an analogous contradiction by interchanging the roles of M and W.

Now, we consider the case of $|M| > 2$ or $|W| > 2$. Since both $|M| \geq 2$ or $|W| \geq 2$, let m_1 and m_2 be distinct members of M and let w_1 and w_2 be distinct members of W. Let $m_3, m_4, \ldots$ be a labeling of $M \setminus \{m_1, m_2\}$ and $w_3, w_4, \ldots$ be a labeling of $W \setminus \{w_1, w_2\}$. We now start with $u \in \mathcal{U}^{slin}$ such that for each $(\pi, z) \in Z$,

$$\begin{aligned}
u_{m_1}(\pi_{m_1}, z_{m_1}) &= K\pi_{m_1w_1} + 0.5K\pi_{m_1w_2} - \textstyle\sum_{t=3}^{|W|} t\pi_{m_1w_t} + z_{m_1}, \\
u_{m_2}(\pi_{m_2}, z_{m_2}) &= K\pi_{m_2w_2} + 0.5K\pi_{m_2w_1} - \textstyle\sum_{t=3}^{|W|} t\pi_{m_2w_t} + z_{m_2}, \\
u_{w_1}(\pi_{w_1}, z_{w_1}) &= K\pi_{w_1m_2} + 0.5K\pi_{w_1m_1} - \textstyle\sum_{t=3}^{|M|} t\pi_{w_1m_t} + z_{w_1}, \\
u_{w_2}(\pi_{w_2}, z_{w_2}) &= K\pi_{w_2m_1} + 0.5K\pi_{w_2m_2} - \textstyle\sum_{t=3}^{|M|} t\pi_{w_2m_t} + z_{w_2},
\end{aligned}$$

for each $m \in M \setminus \{m_1, m_2\}$,

$$u_m(\pi_m, z_m) = z_m - \sum_{t=1}^{|W|} t\pi_{mw_t},$$

and for each $w \in W \setminus \{w_1, w_2\}$,

$$u_w(\pi_w, z_w) = z_w - \sum_{t=1}^{|M|} t\pi_{wm_t}.$$

For any $(\pi, z) \in \mathcal{Z}$, if there are distinct $i, j \in N$ such that $i \notin \{m_1, m_2, w_1, w_2\}$ and $\pi_{ij} > 0$, then decreasing π_{ij} while increasing π_{ii} and π_{jj} by the same amount and maintaining feasibility, ceteris paribus, leads to a Pareto-improvement of (π, z). Thus, if (π, z) is Pareto-efficient, for each $i \in N \setminus \{m_1, m_2, w_1, w_2\}$, $\pi_{ii} = 1$. If (π, z) is also individually rational, then $z_i = \omega_i$. Since each agent in $N \setminus \{m_1, m_2, w_1, w_2\}$ remains unmatched and consumes his or her endowment of money at any Pareto-efficient and individually rational allocation, the proof proceeds in the same manner as the case with $|M| = |W| = 2$. $\square$

Tightness of assumptions We discuss here the assumptions under which we have proven Theorem 1. First is the cardinalities of M and W. If we consider cases where either $|M| = 1$ or $|W| = 1$, then whether strategy-proofness, individual rationality, and Pareto-efficiency are compatible or not depends on the extent to which utility is transferable between agents.

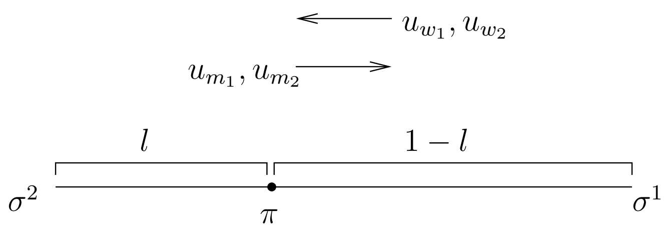
To start with, consider the case where no agent has any endowment of money. That is, for each $i \in N, \omega_i = 0$. If we drop the assumption $|M| \geq 2$ and $|W| \geq 2$, then it is without loss of generality to suppose that $M = \{m\}$. Regardless of cardinality of W, there is an individually rational, Pareto-efficient, and strategy-proof rule φ defined over $\mathcal{U}^{lin}$ as follows: Given $u \in \mathcal{U}^{lin}$, let $\tilde{W} \equiv \{w \in W : u_w(\delta_w^m, 0) \geq u_w(\delta_w^w, 0)$ and $u_m(\delta_m^w, 0) \geq u_m(\delta_m^m, 0)\}$ and, if $\tilde{W} \neq \varnothing$, let $\tilde{w} \in \text{argmax}_{w \in \tilde{W}} u_m(\delta_m^w, 0)$. Finally let, $\varphi(u) \equiv (\pi, 0)$ where π is such that if $\tilde{W} \neq \varnothing$, $\pi_{m\tilde{w}} = 1$ and for each $w \in W \setminus \{\tilde{w}\}, \pi_{ww} = 1$ but if $\tilde{W} = \varnothing$, for each $i \in N, \pi_{ii} = 1$. Thus, the assumption that $|M| \geq 2$ and $|W| \geq 2$ is necessary for Theorem 1 in the absence of any money-endowments.

If at least one agent has a positive endowment of money, then the cardinality assumption is not necessary to obtain the impossibility. Suppose $M = \{m\}$ and $W = \{w\}$ and, without loss of generality, that $\omega_m > 0$. Any rule φ defined over $\mathcal{U}^{slin}$ would be well-defined over the subdomain $\mathcal{U}^{tu}$, where

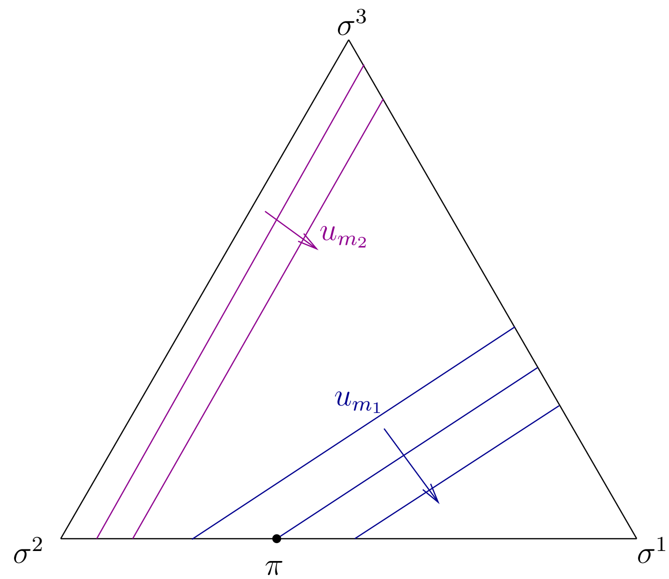
$$\mathcal{U}_m^{tu} \equiv \{u_m \in \mathcal{U}_m^{slin} : \text{ there is } v \in [0, \omega_m] \text{ such that } u_m(\pi_m, z_m) = v\pi_{mw} + z_m\},$$

$$\mathcal{U}_w^{tu} \equiv \{u_w \in \mathcal{U}_w^{slin} : \text{ there is } c \in [-\omega_m, 0] \text{ such that } u_w(\pi_w, z_w) = c\pi_{mw} + z_w\}.$$

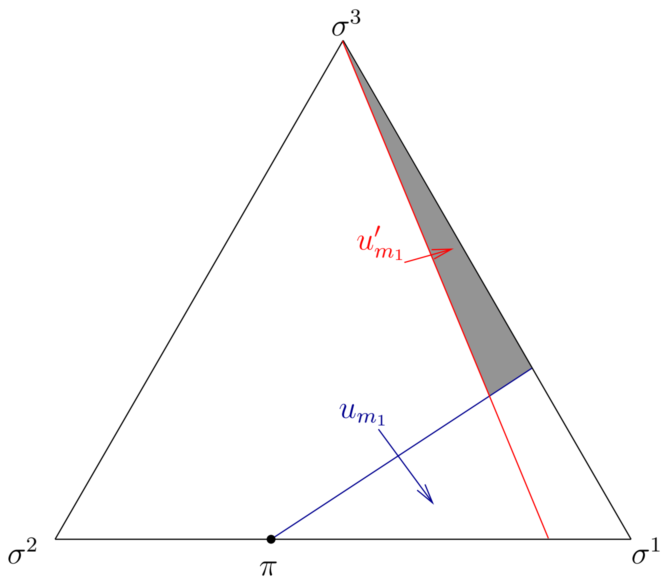
For each $u \in \mathcal{U}^{tu}$, individual rationality and Pareto-efficiency of $(\pi, z) \in \mathcal{Z}$ require that if $v + c > 0$, then $\pi_{mw} = 1$ and if $v + c < 0$, then $\pi_{mw} = 0$. Notice that, having fixed ω, we have



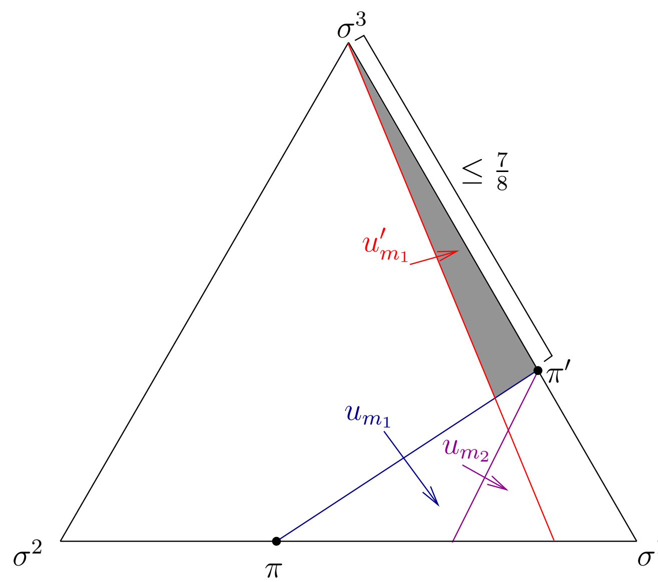
(a) At u, Pareto-efficiency requires π to be a convex combination of σ^1 and σ^2. The figure indicates directions of increasing preferences for each agent. As in the proof of Theorem 1, we consider the case where $l \leq \frac{1}{2}$.



(b) Among convex combinations of σ^1, σ^2, and σ^3, indifference curves of u_{m_1} are flatter than the σ^2-σ^3 face of the simplex since m_1 is fully single at σ^3. On the other hand, indifference curves of u_{m_2} are parallel to it since m_2 is matched fully to w_1 by both σ^2 and σ^3. At (u'_{m_1}, u_{-m_1}), σ^3 is also Pareto-efficient.

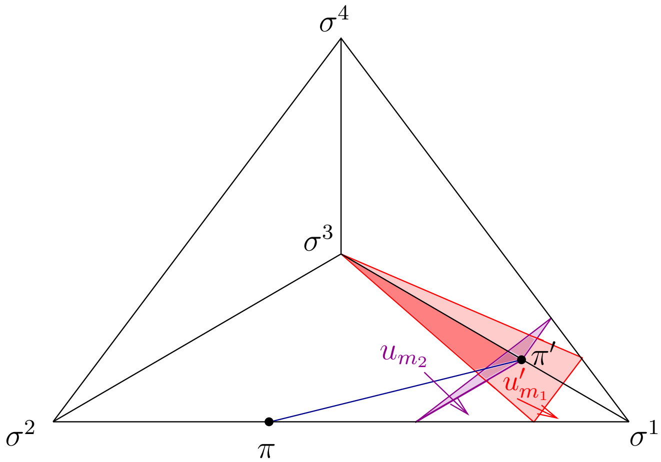


(c) At u'_{m_1}, $\pi = \varphi(u)$ is worse for m_1 than being fully unmatched. Thus, individual rationality requires that $\pi' = \varphi(u_{m_1}, u_{-m_1})$ be to the right of u_{m_1}'s indifference through σ^3. Strategy-proofness requires that it be above (per the orientation of this figure) the indifference curve of u_{m_1} through π, otherwise at the true preference u_{m_1}, m_1 would gain by reporting u'_{m_1}. This leaves the shaded area for π'.

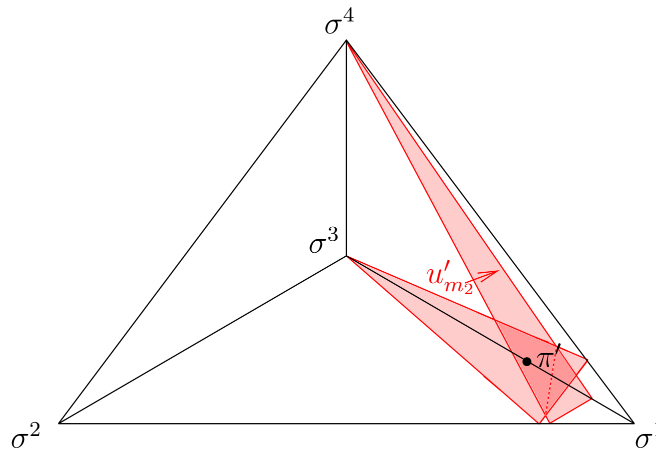


(d) The best choice of π' in terms of u_{m_2} is the one that maximizes the weight on σ^3. The proof proceeds by constructing a profitable misreport for m_2 at (u'_{m_1}, u_{-m_1}), so we mark this optimal π' for u_{m_2}. The weight that π' places on σ^1 is no greater than $\frac{7}{8}$ due to the definition of u_{m_1} and the assumption that $l \leq \frac{1}{2}$.

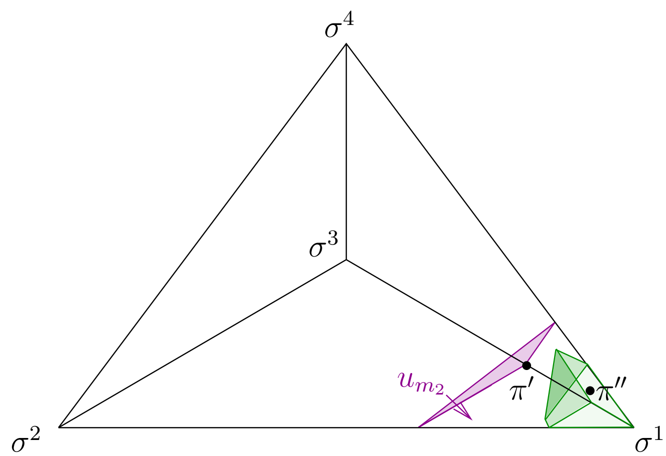
Figure 1: Proof of Theorem 1 for the case of $\omega = 0$.



(a) At (u'_M, u_W), even σ^4 is Pareto-efficient. The indifference planes of u'_{m_1} and u_{m_2} are as indicated above. The indifference plane for u'_{m_1} indicates indifference to shifting mass between σ^2 and σ^4 since both of them match m_1 to w_2 fully.



(b) At u'_{m_2}, π' is worse than being fully unmatched. Individual rationality requires that $\pi'' \equiv \varphi(u'_M, u_W)$ lie above the indifference plane of u'_{m_2} through σ^4, which leaves m_2 fully unmatched.



(c) Individual rationality for m_1 at $u_{m'_1}$ and for m_2 at u'_{m_2} narrow down the possible locations for π'' to the green wedge that includes σ^1. However, every point in this wedge is better according to u_{m_2} than the any choice we could have made for π' in Figure 1d, the best of which is depicted here. This means that φ is not strategy-proof since m_2 gains by reporting u'_{m_2} at the true profile (u'_{m_1}, u_{-m_1}).

Figure 2: Proof of Theorem 1 for the case of $\omega = 0$, continued.

defined $\mathcal{U}^{tu}$ to ensure that money is valuable enough to both agents, relative to a matching's "value" to m or "cost" to w, that it is as though we are in a transferable utility setting. In effect, problems in this subdomain are ones of negotiating efficient trade between a seller and a buyer. These problems have been well studied in the literature and it has long been known that individual rationality, Pareto-efficiency, and strategy-proofness are incompatible for them (Vickrey, 1961; Myerson and Satterthwaite, 1983).

Aside from demonstrating the role of the assumption on $|M|$ and $|W|$, the above discussion touches on another important fact about Theorem 1. If the value of money is adequately high, relative to the value of matchings, then it is as if the model has transferable utility. However, our proof of Theorem 1 proceeds even if the value of money is bounded from above, independently of the endowment. Thus, our result is robust to limits on the transferability of utility.

Independence of axioms For completeness, we show that the axioms in Theorem 1 are independent. If $\omega > 0$, then any rule that selects a *DIP-equilibrium* allocation (Manjunath, 2016) for every profile of preferences is individually rational and Pareto-efficient, but not strategy-proof. If $\omega \not> 0$, then any rule that selects a limit—as $\varepsilon \to 0$—of ε*DIP-equilibrium* allocations is an example of such a rule. The "no trade" rule, which leaves every agent unmatched and does not reallocate money, is individually rational and strategy-proof, but it is not Pareto-efficient. Finally, any serial dictatorship over $\mathcal{Z}$ is Pareto-efficient and strategy-proof, but is not individually rational.

4 Probabilistic Matching

A natural interpretation of our model, particularly when preferences are restricted to $\mathcal{U}^{slin}$, is that of probabilistic matching. Given this interpretation, define the set of *deterministic matchings* as

$$\mathcal{M} \equiv \{\mu \in \Pi : \text{ for each } m \in M \text{ and } w \in W, \pi_{mw} \in \{0,1\}\}.$$

Elements of Π correspond to bistochastic matrices, so by the Birkhoff-von Neumann Theorem, Π is the convex hull of $\mathcal{M}$ (Birkhoff, 1946; von Neumann, 1953). That is, each element of Π corresponds to a probability distribution over $\mathcal{M}$. So, there is a natural *ex post* weakening of the notion of Pareto-efficiency that we have defined in Section 2. We say $(\pi, z) \in \mathcal{Z}$ is *ex post* Pareto-efficient if π can be expressed as a probability distribution over $\mathcal{M}$ such that for each μ in its support, (μ, z) is Pareto-efficient.[7] While every Pareto-efficient allocation is *ex post* Pareto-efficient, the converse is not necessarily true.[8] Our definition of individual rationality is an ex ante concept. In contrast with efficiency, it is weaker than its *ex post* counterpart.

In our proof of Theorem 1, we only appeal to Pareto-efficiency to the extent the Pareto-set is contained in the convex hull of the *ex post* Pareto-set. Thus, weakening the efficiency notion to *ex post* Pareto-efficiency does not permit us to escape from the impossibility of Theorem 1.

Proposition 1 *If $|M| \geq 2$ and $|W| \geq 2$, then no strategy-proof rule for $\mathcal{U}^{slin}$ selects an individually rational and ex post Pareto-efficient allocation for every probabilistic allocation problem.*

Ordinal rules in the absence of money When $\omega = 0$, a potential restriction that we may impose on a rule is that it be invariant to all information other than how agents rank their possible partners. A rule φ with domain $\mathcal{U} \subseteq \mathcal{U}^{slin}$ is *ordinal* if it only depends on induced preferences over partners. That is, for every pair $u, u' \in \mathcal{U}$ such that for each $i \in N$, $R(u_i) = R(u'_i)$, $\varphi(u) = \varphi(u')$. A rule is *deterministic* if for each $u \in \mathcal{U}$, $\varphi(u) = (\mu, 0)$ such that $\mu \in \mathcal{M}$.[9]

Alcalde and Barberà (1994) show that no strategy-proof, ordinal, and deterministic rule selects an individually rational and *ex post* Pareto-efficient for every problem. The following corollary of Proposition 1 strengthens this by showing the impossibility even when we permit randomization.

Corollary 1 *If $|M| \geq 2$ and $|W| \geq 2$, then no strategy-proof and ordinal rule for $\mathcal{U}^{slin}$ selects an individually rational and ex post Pareto-efficient matching for every probabilistic matching problem without money.*

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[7]We have supposed that the money component of the allocation is not randomized. However, given that we are interested in an *ex ante* notion of individual rationality and preferences are linear in money, this is without loss of generality.

[8]This can be established similarly to the case where one side, either M or W, are indifferent between all partners as in the probabilistic object allocation problem (Bogomolnaia and Moulin, 2001).

[9]Rules, as we have defined them, select elements of $\mathcal{Z}$. For consistency, we have retained the money component of an allocation even though it is necessarily zero since $\omega = 0$.

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