The Impossibility of Strategy-proof, Pareto-efficient, and Individually Rational Rules for Fractional Matching*

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Abstract

For a model of fractional matching with linear utility, interpreted as probabilistic matching, we show that strategy-proofness, ex post Pareto-efficiency, and ex ante individual rationality are incompatible. This result is robust to whether, or to what extent, transfers are possible. Since we prove this impossibility for the domain of preferences with linear utility representations and since ex post Pareto-efficiency is weaker than Pareto-efficiency from the ex ante perspective, it implies an incompatibility for more general fractional matching applications as well.

JEL classification: C71, C78, D51

Keywords: *fractional, matching, strategy-proofness, Pareto-efficiency, individual rationality*

1 Introduction

We study a model of fractional matching with money (Roth et al., 1993; Manjunath, 2016). Each agent has a unit of availability to be matched with potential partners or remain alone. Each agent is also endowed with a non-negative quantity—possibly zero—of a perfectly divisible good that we call money. A natural interpretation of this model is as one of probabilistic matching. For the domain of linear, that is von Neumann-Morgenstern, utilities that are increasing in money, we show that strategy-proofness,[1] ex post Pareto-efficiency,[2]

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[1]Strategy-proofness means that no agent receives a preferable, from an ex ante perspective, assignment by misreporting his or her preferences.

[2]Ex post Pareto-efficiency means that for each deterministic matching that can be realized, no agent can be made better off without making another agent worse off.

and ex ante individually rationality[3] are incompatible.

For interpretations other than probabilistic matching, the preference domain may be larger than the linear utilities that we consider. Furthermore, the appropriate formulation of Pareto-efficiency may correspond to what would be *ex ante* Pareto-efficiency for probabilistic matching. Our result extends to such settings since not only is ex post Pareto-efficiency weaker than ex ante Pareto-efficiency, but also the more restrictive assumption on preferences strengthens the impossibility.

For the deterministic marriage (and roommate) problem (Gale and Shapley, 1962), no individually rational and Pareto-efficient rule is strategy-proof (Alcalde and Barberà, 1994; Sönmez, 1999). Such a problem can be represented as a fractional matching problem where no agent is endowed with a positive amount of money, allocations are deterministic, and preferences over partners are strict and have a linear utility representation. Our impossibility result improves on the impossibility result of Alcalde and Barberà (1994) by dropping the requirements that the rule be deterministic and only use ordinal preference information. While we have assumed that all preferences with strict orderings over partners are available, our proof technique only requires that it be possible for each agent to have two indifference classes among acceptable partners. If agents are indifferent between all acceptable partners, that is if preferences are *dichotomous*, then strategy-proofness, *ex ante* Pareto-efficiency, and *ex post* individual rationality are indeed compatible (Bogomolnaia and Moulin, 2004).

For the probabilistic allocation of objects without money, strategy-proofness is incompatible with Pareto-efficiency and equal treatment of equals (Zhou, 1990).[4,5] This problem may be encoded in our model by assuming that agents on one side are indifferent between all partners. However, our impossibility does not hold for this special case: every serial dictatorship is strategy-proof, *ex ante* Pareto-efficient, and *ex post* individually rational. Indeed, the set of Pareto-efficient deterministic allocations expands greatly as we take the limit of strict preferences for both sides tending towards one side having complete indifference. Thus, results shown on this limiting domain have little bearing on the general model that we study.

The difficulty with obtaining a strategy-proof, Pareto-efficient, and individually rational rule is a familiar one in other economic settings. Such rules do not exist for exchange

[3] Ex ante individual rationality means that no agent finds remaining single and consuming his or her endowment of money preferable to his or her fractional assignment. This is weaker than *ex post* individual rationality, which requires that each agent prefer each realized allocation to remaining alone and consuming his or her endowment of money.

[4] With comparisons based on stochastic dominance, the corresponding notions of efficiency, fairness, and strategy-proofness are incompatible (Bogomolnaia and Moulin, 2001; Nesterov, 2017) even if the domain of ordinal preferences is restricted (Kasajima, 2013; Chang and Chun, 2016).

[5] For the allocation of objects *with* money, as pointed out by Schummer (2000), whether strategy-proofness and Pareto-efficiency together imply dictatorship remains an open question.

economies with classical preferences over at least two divisible private goods (Hurwicz, 1972; Serizawa, 2002; Momi, 2017). Even for exchange economies with one or more public goods produced linearly from one private good, and for social choice with lotteries over alternatives, strategy-proofness and Pareto-efficiency lead to dictatorial rules (Schummer, 1999). On the surface, these results may appear related to ours: the space of lotteries over social alternatives is a simplex, as is a fractional matching, and preferences have linear representations. Also, a fractional matching between two agents bears resemblance to a public good, its consumption being non-rival between the pair. However, our result differs in important ways: (1) We appeal to ex post rather than ex ante Pareto-efficiency. (2) The feasible set in our model is the product of two simplices, a fractional matching and a division of the available money, and therefore not itself a simplex. (3) We restrict attention to the set of linear utilities such that each agent is indifferent between two corners of the matching simplex if and only if they assign him the same mate, as opposed to the set of all possible linear utilities. (4) Interpreting fractional match between a pair as a public good, each agent's availability would have to be a private good that has no value to any other agent, except as an input into a Leontieff, as opposed to linear, technology that produces the match. So, to the best of our knowledge, none of the existing impossibility results about strategy-proof and efficient rules for economies with private or public goods covers ours.

2 The Model

A *fractional matching model* consists of a finite set of agents N. Assume for now N is partitioned into two non-empty sets: M and W.

Each agent has unit availability. For each $m \in M$, a *fractional allocation* divides his availability between partners in W and being alone. That is, a fractional allocation for m is

$$\pi_m \in \Delta_m \equiv \left\{ y_m \in \mathbb{R}_+^{W \cup \{m\}} : \sum_{i \in W \cup \{m\}} y_{mi} = 1 \right\}.$$

For each $w \in W$, π_{mw} represents the amount that m is matched to w while π_{mm} is the amount he remains alone. For each $w \in W$, define Δ_w similarly and let $\Delta \equiv \times_{i \in N} \Delta_i$. For each $i \in N$ and each possible partner j for i, let $\delta_i^j \in \Delta_i$ be such that $\delta_{ij}^j = 1$. We define δ_i^i analogously.

Consider a profile of fractional allocations, $\pi \in \Delta$. For each $m \in M$ and $w \in W$, π_{mw} is the amount that m is matched to w and π_{wm} is the amount that w is matched with m. For this to make any sense, these have to be the same: the amount they are matched to one another. So feasibility requires that $\pi_{mw} = \pi_{wm}$. Such a π is a *fractional matching.* Let Π

be the set of all fractional matchings.

A natural interpretation of this model is that of probabilistic matching. Given this interpretation, define the set of *deterministic matchings* as

$$\Sigma \equiv \{\sigma \in \Pi : \text{ for each } m \in M \text{ and } w \in W, \sigma_{mw} \in \{0,1\}\}.$$

Elements of Π correspond to bistochastic matrices, so by the Birkhoff-von Neumann Theorem, Π is the convex hull of Σ. That is, each element of Π can be expressed as a probability distribution over Σ.

Aside from being matched to one another, agents also consume a divisible private good that we call money. Each $i \in N$ is endowed with $\omega_i \in \mathbb{R}_+$ units of money. Since i consumes a fractional allocation along with an amount of money, his consumption set is $X_i \equiv \Delta_i \times \mathbb{R}_+$. Let $X \equiv \underset{i \in N}{\times} X_i$. This includes the possibility that for each $i \in N$, $\omega_i = 0$ so there are no transfers between agents.

An *allocation* is $(\pi, z) \in X$ such that π is a fractional matching and z feasibly allocates all of the money in the problem. That is, $\pi \in \Pi$ and $\sum_{i \in N} z_i = \sum_{i \in N} \omega_i$. Let $\mathcal{Z}$ be the set of allocations.

For each $i \in N$, i's preference over X_i is described by a utility function $u_i : X_i \to \mathbb{R}$. Let $\tilde{\mathcal{U}}_i$ be the set of all such utility functions and let $\tilde{\mathcal{U}} \equiv \times_{i \in N} \tilde{\mathcal{U}}_i$.

Since we fix M, W, and ω, a *problem* is fully described by a profile of utility functions $u \in \tilde{\mathcal{U}}$. A *domain* of problems, $\mathcal{U}$, is a Cartesian product subset of $\tilde{\mathcal{U}}$. A *rule for* $\mathcal{U}$, $\varphi : \mathcal{U} \to \mathcal{Z}$, assigns to each problem u in $\mathcal{U}$ a feasible allocation $\varphi(u) \in \mathcal{Z}$.

Linear Utility For each $i \in N$, let $\mathcal{U}_i^{lin}$ be the set of all linear utility functions in $\tilde{\mathcal{U}}_i$ that are increasing in money. Under the interpretation of a $\pi \in \Pi$ as a probability distribution, these utility functions represent von Neumann-Morgenstern preferences over the matching component of an allocation. For each $u_i \in \mathcal{U}_i^{lin}$, let $R(u_i)$ be the preference relation over i's partners induced by u_i. That is, if $i \in M$, let $R(u_i)$ be an ordering over $W \cup \{i\}$ such that for each pair $j, k \in W \cup \{i\}$, $j \; R(u_i) \; k$ if and only if $u_i(\delta_i^j, 0) \geq u_i(\delta_i^k, 0)$, and similarly if $i \in W$. Let $\mathcal{U}_i^{slin}$ be the set of all linear utility functions for i that induce *strict* preferences over i's partners. That is, for each $u_i \in \mathcal{U}_i^{slin}$, $R(u_i)$ is a linear order over i's partners.

In showing our results, we restrict attention to $\mathcal{U}^{slin} \equiv \underset{i \in N}{\times} \mathcal{U}_i^{slin}$, the subdomain of problems with linear utility that are increasing in money *and* induce strict preferences over partners.

Properties of Allocations and Rules We first define properties that are meaningful for all problems in $\tilde{\mathcal{U}}$.

Let $u \in \tilde{\mathcal{U}}$. If $(\pi, z) \in \mathcal{Z}$ is such that for each $i \in N$, $u_i(\pi_i, z_i) \geq u_i(\delta_i^i, \omega_i)$, then (π, z) is *ex ante individually rational at* u. If there is no other $(\pi', z') \in \mathcal{Z}$ such that for each $i \in N$, $u_i(\pi'_i, z'_i) \geq u_i(\pi_i, z_i)$ and for some $i \in N$, $u_i(\pi'_i, z'_i) > u_i(\pi_i, z_i)$, then (π, z) is *ex ante Pareto efficient at* u.

Given a domain of problems, $\mathcal{U} \subseteq \tilde{\mathcal{U}}$, a rule $\varphi : \mathcal{U} \to \mathcal{A}$ is *strategy-proof* if, for each $u \in \mathcal{U}$ and each $i \in N$, there is no $u'_i \in \mathcal{U}_i$ such that

$$u_i(\varphi_i(u'_i, u_{-i})) > u_i(\varphi_i(u)).$$

We now define properties that rely on the interpretation of fractional matchings as probability distributions with preferences drawn from $\mathcal{U}^{slin}$. Let $u \in \mathcal{U}^{slin}$. Each $\sigma \in \Sigma$ is *deterministically individually rational* if, for each $i \in N$, $u_i(\sigma_i) \geq u_i(\delta_i^i)$. If $(\pi, z) \in \mathcal{Z}$ is such that π a convex combination of deterministically individually rational matchings, then (π, z) is *ex post individually rational.* Every ex post individually rational allocation is ex ante individually rational, but that the converse is not true.[6]

Each $\sigma \in \Sigma$ is *deterministically Pareto-efficient* if there is no σ' such that for each $i \in N$, $u_i(\sigma'_i) \geq u_i(\sigma_i)$ and for some $i \in N$, $u_i(\sigma'_i) > u_i(\sigma_i)$. If $(\pi, z) \in \mathcal{Z}$ is such that π a convex combination of deterministically Pareto-efficient matchings, then (π, z) is *ex post Pareto-efficient.* Every ex ante Pareto-efficient allocation is ex post Pareto-efficient, but the converse is not true.[7]

3 Impossibility

The following result holds for fractional matching problems with or without money. In the proof, we have only relied upon linear utility functions that induce strict preferences over partners. We have also imposed the weaker versions of individual rationality and Pareto-efficiency. That we have shown an incompatibility between the weaker axioms on such a small domain implies that it is particularly robust: it holds for any domain that contains $\mathcal{U}^{slin}$. The proof is also invariant to the amount of money in the problem.

We illustrate the proof in Figures 1 and 2 for the case without money—where, for each

[6]We have supposed that the money component of the allocation is not randomized. However, given that we are interested in an *ex ante* notion of individual rationality and preferences are linear in money, this is without loss of generality.

[7]This can be established similarly to the case where agents on one side, either M or W, are indifferent between all partners as in the probabilistic object allocation problem (Bogomolnaia and Moulin, 2001).

$i \in N$, $\omega_i = 0$. When there is money, the need for an additional (fourth) dimension makes a diagrammatic representation infeasible.

Theorem 1 *If $|M| \geq 2$, $|W| \geq 2$, and $\mathcal{U} \supseteq \mathcal{U}^{slin}$, then no rule for $\mathcal{U}$ is strategy-proof, ex post Pareto-efficient, and ex ante individually rational.*

Proof: We start with the case where $|M| = |W| = 2$. Let $M \equiv \{m_1, m_2\}$ and $W \equiv \{w_1, w_2\}$. Let $\Omega \equiv \sum_{i \in N} \omega_i \geq 0$, $K > 12\Omega$, and $\alpha > \frac{14K+32\Omega}{K-12\Omega}$. By definition of Ω and K, $\frac{14K+32\Omega}{K-12\Omega} \geq 14$, and so $\alpha > 14$.[8,9]

Let $u \in \mathcal{U}^{slin}$ be such that for each $(\pi, z) \in \mathcal{Z}$,

$$\begin{aligned}
u_{m_1}(\pi_{m_1}, z_{m_1}) &= K\pi_{m_1w_1} + \tfrac{K}{2}\pi_{m_1w_2} + z_{m_1},\\
u_{m_2}(\pi_{m_2}, z_{m_2}) &= K\pi_{m_2w_2} + \tfrac{K}{2}\pi_{m_2w_1} + z_{m_2},\\
u_{w_1}(\pi_{w_1}, z_{w_1}) &= K\pi_{w_1m_2} + \tfrac{K}{2}\pi_{w_1m_1} + z_{w_1}, \text{ and}\\
u_{w_2}(\pi_{w_2}, z_{w_2}) &= K\pi_{w_2m_1} + \tfrac{K}{2}\pi_{w_2m_2} + z_{w_2}.
\end{aligned}$$

Suppose that φ is a strategy-proof, ex post Pareto-efficient, and ex ante individually rational rule defined on $\mathcal{U}$. Let $(\pi, z) \equiv \varphi(u)$. By definition of u, the only deterministically Pareto-efficient matchings are σ^1 and $\sigma^2 \in \Sigma$ where $\sigma^1_{m_1w_1} = \sigma^1_{m_2w_2} = 1$ and $\sigma^2_{m_1w_2} = \sigma^2_{m_2w_1} = 1$. By ex post Pareto-efficiency, π is a convex combination of σ^1 and σ^2. Let $l \in [0,1]$ be such that $\pi = l\sigma^1 + (1-l)\sigma^2$.

Suppose that $l \leq \frac{1}{2}$. Let $u'_{m_1} \in \mathcal{U}^{slin}_{m_1}$ be such that for each $(\pi_{m_1}, z_{m_1}) \in X_{m_1}$,

$$u'_{m_1}(\pi_{m_1}, z_{m_1}) = K\pi_{m_1w_1} - \alpha K\pi_{m_1w_2} + z_{m_1}.$$

Let $(\pi', z') \equiv \varphi(u'_{m_1}, u_{-m_1})$. By definition of (u_{m_1}, u_{-m_1}), the only deterministically Pareto-efficient matchings are σ^1 and σ^2 defined above and $\sigma^3 \in \Sigma$ such that $\sigma^3_{m_1m_1} = \sigma^3_{w_2w_2} = \sigma^3_{m_2w_1} = 1$. By ex post Pareto-efficiency, π' is a convex combination of σ^1, σ^2, and σ^3. Thus,

$$\pi'_{m_1w_1} = \pi'_{m_2w_2}. \tag{1}$$

Notice that $u_{m_1}(\pi_{m_1}, z_{m_1}) = Kl + K(\frac{1-l}{2}) + z_{m_1} = K(\frac{1+l}{2}) + z_{m_1}$. By strategy-proofness,

$$K\pi'_{m_1w_1} + \frac{K}{2}\pi'_{m_1w_2} + z'_{m_1} = u_{m_1}(\pi'_{m_1}, z'_{m_1}) \leq u_{m_1}(\pi_{m_1}, z_{m_1}) = K\left(\frac{1+l}{2}\right) + z_{m_1}.$$

[8]We define α this way to ensure that $\frac{\alpha K - \Omega}{\alpha+2} > \frac{7}{8}K + \frac{3}{2}\Omega$, a fact that we appeal to below.

[9]Since Figures 1 and 2 demonstrate the proof where $\Omega = 0$, K is any positive number and $\alpha > 14$.

Then, since $K > 0$, $z_{m_1} \leq \Omega$, and $z'_{m_1} \geq 0$,

$$K\pi'_{m_1w_1} \leq K\pi'_{m_1w_1} + K\pi'_{m_1w_2} \leq K\left(\frac{1+l}{2}\right) + z_{m_1} - z'_{m_1} \leq K\left(\frac{1+l}{2}\right) + \Omega.$$

Thus, since $l \leq \frac{1}{2}$ and $K > 12\Omega$,

$$\pi'_{m_1w_1} \leq \frac{3}{4} + \frac{\Omega}{K} < 1. \tag{2}$$

Since $\pi'_{m_2m_2} = 0$ and $z'_{m_2} \leq \Omega$, by (1) and (2),

$$u_{m_2}(\pi'_{m_2}, z'_{m_2}) \leq K\left(\frac{3}{4} + \frac{\Omega}{K}\right) + \frac{K}{2}\left(\frac{1}{4} - \frac{\Omega}{K}\right) + \Omega = \frac{7}{8}K + \frac{3}{2}\Omega. \tag{3}$$

Let $u'_{m_2} \in \mathcal{U}^{slin}_{m_2}$ be such that for each $(\pi_{m_2}, z_{m_2}) \in X_{m_2}$,

$$u'_{m_2}(\pi_{m_2}, z_{m_2}) = K\pi_{m_2w_2} - \alpha K\pi_{m_2w_1} + z_{m_2}.$$

Define $(\pi'', z'') \equiv \varphi(u'_M, u_W)$. By definition of (u'_M, u_W), the only deterministically Pareto-efficient matchings are σ^1, σ^2, and σ^3 defined above and σ^4 such that $\sigma^4_{m_1w_2} = \sigma^4_{m_2m_2} = \sigma^4_{w_1w_1} = 1$. By ex post Pareto-efficiency, π'' is a convex combination of $\sigma^1, \sigma^2, \sigma^3$, and σ^4. So there are $p, q, r, s \in [0,1]$ such that $\pi'' = p\sigma^1 + q\sigma^2 + r\sigma^3 + s\sigma^4$ and $p + q + r + s = 1$.

Since (π'', z'') is individually rational at (u'_M, u_W) and $p = 1 - q + r + s$,

$$u'_{m_1}(\pi''_{m_2}, z''_{m_1}) = Kp - \alpha K(q+s) + z''_{m_1} \geq \omega_{m_1} = u'_{m_1}(\delta^{m_1}_{m_1}, \omega_{m_1}) \tag{4}$$

and

$$u'_{m_2}(\pi''_{m_2}, z''_{m_2}) = Kp - \alpha K(q+r) + z''_{m_2} \geq \omega_{m_2} = u''_{m_2}(\delta^{m_2}_{m_2}, \omega_{m_2}) \tag{5}$$

Adding (4) and (5),

$$2Kp - \alpha K(q+r+s) - \alpha Kq + z''_{m_1} + z''_{m_2} \geq \omega_{m_1} + \omega_{m_2},$$

which, since $p = 1 - q - r - s$, implies

$$2Kp - \alpha K(1-p) \geq \alpha Kq + \omega_{m_1} + \omega_{m_2} - z''_{m_1} - z''_{m_2}.$$

Then, since $\alpha K q \geq 0$ and $\omega_{m_1} + \omega_{m_2} - z''_{m_1} - z''_{m_2} \geq -\Omega$, we have $2Kp - \alpha K(1-p) \geq -\Omega$, so

$$p \geq \frac{\alpha K - \Omega}{(\alpha+2)K}. \tag{6}$$

By definition of K and α, $\alpha K > \Omega$. Thus, the right hand side of (6) is positive but less than one.

Finally, by (6),

$$u_{m_2}(\pi''_{m_2}, z''_{m_2}) = Kp + \frac{K}{2}(q+r) + z''_{m_2} \geq Kp \geq \frac{\alpha K - \Omega}{\alpha + 2}.$$

However, by definition of α and (3),

$$u_{m_2}(\pi''_{m_2}, z''_{m_2}) \geq \frac{\alpha K - \Omega}{\alpha + 2} > \frac{7}{8}K + \frac{3}{2}\Omega \geq u_{m_2}(\pi'_{m_2}, z'_{m_2}).$$

This contradicts the strategy-proofness of φ, since m_2 can profitably manipulate it by reporting u'_{m_2} at the profile (u'_{m_1}, u_{-m_1}).

Thus, $l > \frac{1}{2}$. However, we reach an analogous contradiction by interchanging the roles of M and W.

Now, we consider the case of $|M| > 2$ or $|W| > 2$. Since both $|M| \geq 2$ or $|W| \geq 2$, let m_1 and m_2 be distinct members of M and let w_1 and w_2 be distinct members of W. Let $m_3, m_4, \ldots$ be a labeling of $M \setminus \{m_1, m_2\}$ and $w_3, w_4, \ldots$ be a labeling of $W \setminus \{w_1, w_2\}$. We start with $u \in \mathcal{U}^{slin}$ such that for each $(\pi, z) \in Z$,

$$\begin{array}{rcl}
u_{m_1}(\pi_{m_1}, z_{m_1}) & = & K\pi_{m_1w_1} + \frac{K}{2}\pi_{m_1w_2} - \sum_{t=3}^{|W|} t\pi_{m_1w_t} + z_{m_1}, \\
u_{m_2}(\pi_{m_2}, z_{m_2}) & = & K\pi_{m_2w_2} + \frac{K}{2}\pi_{m_2w_1} - \sum_{t=3}^{|W|} t\pi_{m_2w_t} + z_{m_2}, \\
u_{w_1}(\pi_{w_1}, z_{w_1}) & = & K\pi_{w_1m_2} + \frac{K}{2}\pi_{w_1m_1} - \sum_{t=3}^{|M|} t\pi_{w_1m_t} + z_{w_1}, \\
u_{w_2}(\pi_{w_2}, z_{w_2}) & = & K\pi_{w_2m_1} + \frac{K}{2}\pi_{w_2m_2} - \sum_{t=3}^{|M|} t\pi_{w_2m_t} + z_{w_2},
\end{array}$$

for each $m \in M \setminus \{m_1, m_2\}$,

$$u_m(\pi_m, z_m) = z_m - \sum_{t=1}^{|W|} t\pi_{mw_t},$$

and for each $w \in W \setminus \{w_1, w_2\}$,

$$u_w(\pi_w, z_w) = z_w - \sum_{t=1}^{|M|} t\pi_{wm_t}.$$

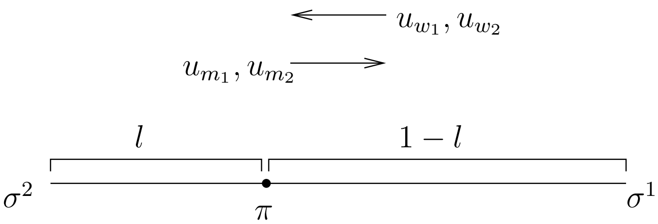
By definition of u, for each deterministically Pareto-efficient matching σ and each distinct pair $i, j \in N$ such that $i \notin \{m_1, m_2, w_1, w_2\}$, $\sigma_{ij} = 0$. Thus, for each ex post Pareto-efficient $(\pi, z) \in \mathcal{Z}$ and each $i \in N \setminus \{m_1, m_2, w_1, w_2\}$, $\pi_{ii} = 1$. If (π, z) is also ex ante individually rational, then $z_i = \omega_i$. Since each agent in $N \setminus \{m_1, m_2, w_1, w_2\}$ remains unmatched and consumes his or her endowment of money at any ex post Pareto-efficient and ex ante individually rational allocation, the proof proceeds in the same manner as the case with $|M| = |W| = 2$. □

Independence of axioms For completeness, we show that the axioms in Theorem 1 are independent, by demonstrating the existence of a rule satisfying every pair from the triplet of axioms. If $\omega > 0$, then any rule that selects a *DIP-equilibrium* allocation (Manjunath, 2016) for every profile of preferences is ex ante (and hence ex post) Pareto-efficient and ex ante individually rational but not strategy-proof. If $\omega \ngtr 0$, then any rule that selects a limit—as $\varepsilon \to 0$—of ε*DIP-equilibrium* allocations is an example of such a rule. The "no trade" rule, which leaves every agent unmatched and does not reallocate money, is strategy-proof and ex ante individually rational, but not ex post Pareto-efficient. Finally, any serial dictatorship over $\mathcal{Z}$ is strategy-proof and ex ante (and therefore ex post) Pareto-efficient but not ex ante individually rational.

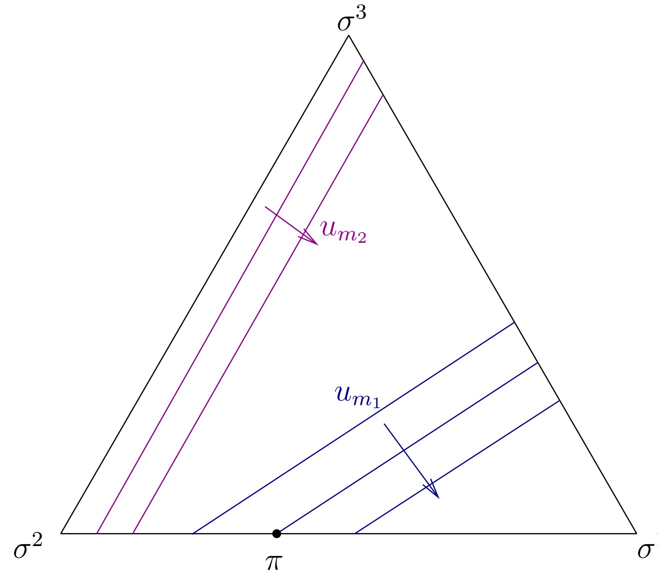
Number of agents and transferability of utility If we consider cases where either $|M| = 1$ or $|W| = 1$, then whether strategy-proofness, ex post Pareto-efficiency, and ex ante individual rationality are compatible or not depends on the extent to which utility is transferable between agents.

To start with, consider the case where no agent has any endowment of money. That is, for each $i \in N, \omega_i = 0$. If either $|M| = 1$ or $|W| = 1$, then it is without loss of generality to suppose that $M = \{m\}$. Regardless of cardinality of W, there is a strategy-proof, ex post Pareto-efficient, and ex ante individually rational rule φ defined over $\mathcal{U}^{lin}$ as follows: Given $u \in \mathcal{U}^{lin}$, let $\tilde{W} \equiv \{w \in W : u_w(\delta^m_w, 0) \geq u_w(\delta^w_w, 0)$ and $u_m(\delta^w_m, 0) \geq u_m(\delta^m_m, 0)\}$ and, if $\tilde{W} \neq \varnothing$, let $\tilde{w} \in \mathrm{argmax}_{w \in \tilde{W}} u_m(\delta^w_m, 0)$. Finally let, $\varphi(u) \equiv (\sigma, 0)$ where $\sigma \in \Sigma$ is such that if $\tilde{W} \neq \varnothing$, $\sigma_{m\tilde{w}} = 1$ but if $\tilde{W} = \varnothing$, for each $i \in N, \sigma_{ii} = 1$. Thus, the assumption that $|M| \geq 2$ and $|W| \geq 2$ is necessary for Theorem 1 in the absence of any money-endowments.

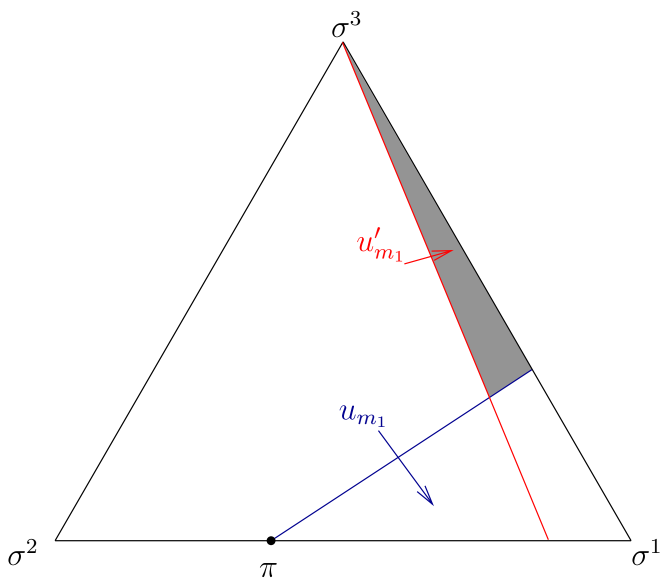
If at least one agent has a positive endowment of money, then the cardinality assumption is not necessary to obtain the impossibility. Suppose $M = \{m\}$ and $W = \{w\}$ and, without loss of generality, that $\omega_m > 0$. Any rule φ defined over $\mathcal{U}^{slin}$ would be well-defined over the



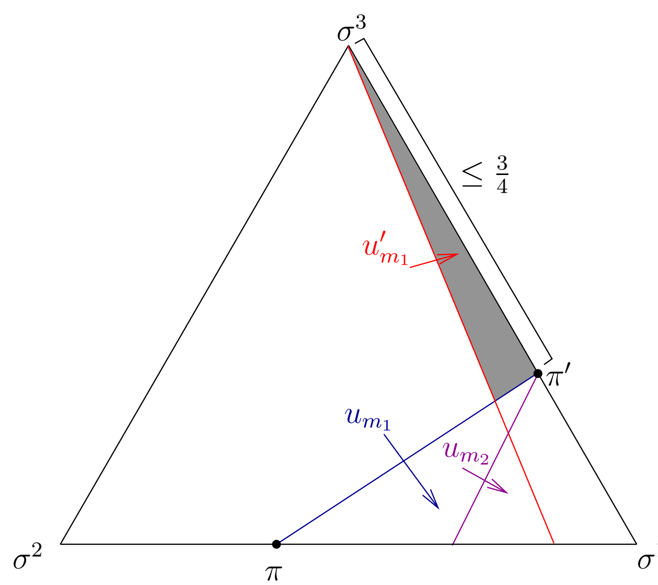
(a) At u, ex post Pareto-efficiency requires π to be a convex combination of σ^1 and σ^2. The figure indicates directions of increasing preferences for each agent. As in the proof of Theorem 1, we consider the case where $l \leq \frac{1}{2}$.



(b) Among convex combinations of σ^1, σ^2, and σ^3, indifference curves of u_{m_1} are flatter than the σ^2-σ^3 face of the simplex since m_1 is fully single at σ^3. On the other hand, indifference curves of u_{m_2} are parallel to it since m_2 is matched fully to w_1 by both σ^2 and σ^3. At (u'_{m_1}, u_{-m_1}), σ^3 is also ex post Pareto-efficient.

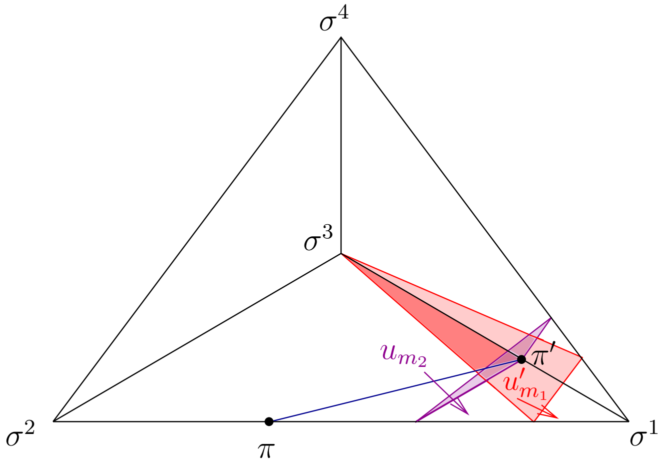


(c) At u'_{m_1}, $\pi = \varphi(u)$ is worse for m_1 than being fully unmatched. Thus, ex ante individual rationality requires that $\pi' = \varphi(u_{m_1}, u_{-m_1})$ be to the right of u_{m_1}'s indifference through σ^3. Strategy-proofness requires that it be above (per the orientation of this figure) the indifference curve of u_{m_1} through π, otherwise at the true preference u_{m_1}, m_1 would gain by reporting u'_{m_1}. This leaves the shaded area for π'.

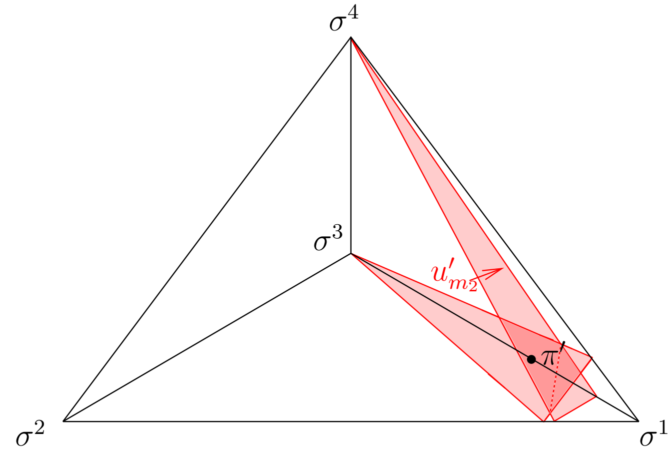


(d) The best choice of π' in terms of u_{m_2} is the one that maximizes the weight on σ^1. The proof proceeds by constructing a profitable misreport for m_2 at (u'_{m_1}, u_{-m_1}), so we mark this optimal π' for u_{m_2}. The weight that π' places on σ^1 is no greater than $\frac{3}{4}$ due to the definition of u_{m_1} and the assumption that $l \leq \frac{1}{2}$.

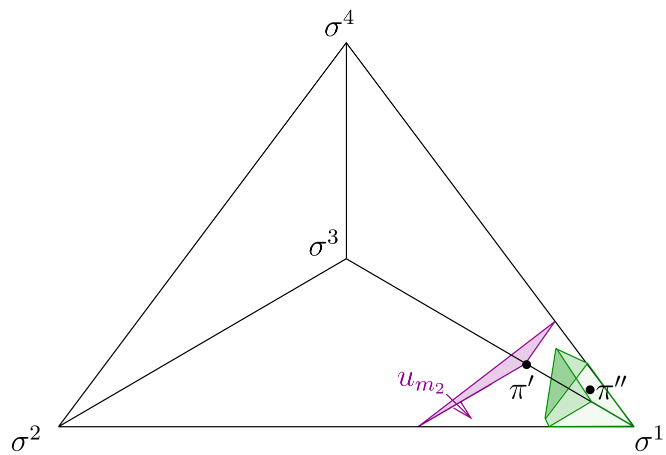
Figure 1: Proof of Theorem 1 for the case without money.



(a) At (u'_M, u_W), even σ^4 is ex post Pareto-efficient. The indifference planes of u'_{m_1} and u_{m_2} are as indicated above. The indifference plane for u'_{m_1} indicates indifference to shifting mass between σ^2 and σ^4 since both of them match m_1 to w_2 fully.



(b) At u'_{m_2}, π' is worse than being fully unmatched. Ex ante individual rationality requires that $\pi'' \equiv \varphi(u'_M, u_W)$ lie above the indifference plane of u'_{m_2} through σ^4, which leaves m_2 fully unmatched.



(c) Ex ante individual rationality for m_1 at $u_{m'_1}$ and for m_2 at u'_{m_2} narrow down the possible locations for π'' to the green wedge that includes σ^1. However, every point in this wedge is better according to u_{m_2} than the any choice we could have made for π' in Figure 1d, the best of which is depicted here. This means that φ is not strategy-proof since m_2 gains by reporting u'_{m_2} at the true profile (u'_{m_1}, u_{-m_1}).

Figure 2: Proof of Theorem 1 for the case without money, continued.

subdomain $\mathcal{U}^{tu}$, where

$$\mathcal{U}_m^{tu} \equiv \{u_m \in \mathcal{U}_m^{slin} : \text{ there is } v \in [0, \omega_m] \text{ such that } u_m(\pi_m, z_m) = v\pi_{mw} + z_m\},$$

$$\mathcal{U}_w^{tu} \equiv \{u_w \in \mathcal{U}_w^{slin} : \text{ there is } c \in [-\omega_m, 0] \text{ such that } u_w(\pi_w, z_w) = c\pi_{mw} + z_w\}.$$

Note that ex post and ex ante Pareto-efficiency are equivalent on this subdomain. For each $u \in \mathcal{U}^{tu}$, (ex post) Pareto-efficiency and ex ante individual rationality of $(\pi, z) \in \mathcal{Z}$ require that if $v + c > 0$, then $\pi_{mw} = 1$ and if $v + c < 0$, then $\pi_{mw} = 0$. Having fixed ω, we have defined $\mathcal{U}^{tu}$ to ensure that money is valuable enough to both agents—relative to a matching's "value" to m or "cost" to w—that it is as though we are in a transferable utility setting. In effect, problems in this domain are ones of negotiating efficient trade between a seller and a buyer. These problems have been well studied in the literature and it has long been known that strategy-proofness, ex post Pareto-efficiency, and ex ante individual rationality are incompatible for them (Vickrey, 1961)

Aside from demonstrating the role of the assumption on $|M|$ and $|W|$, the above discussion touches on another important fact about Theorem 1. If the value of money is adequately high, relative to the value of matchings, then it is as if the model has transferable utility, since the non-negativity constraint on money consumption is not restrictive. However, our proof of Theorem 1 proceeds even if the value of money is bounded from above, independently of the endowment. Thus, our result is robust to limits on the transferability of utility.

Bipartite structure Suppose that we drop this assumption so that any agent can be matched to any other. If $|N| = 2$, then the problem is inherently bipartite and our above discussion is applicable. If $|N| \geq 4$, then the bipartite problem that we have considered is, essentially, a subdomain: those preference profiles where the set of agents is partitioned into two groups such that each agent derives positive utility from members of the other group but negative utility from members of his own group. On this subdomain, ex post Pareto-efficiency rules out any matching of agents in the same group and so Theorem 1 applies to the larger domain of non-bipartite problems.

We turn, finally, to the case of $|N| = 3$. As described above, problems with the bipartite structure may be embedded in the more general domain without this structure. However, if there are only three agents, then one of the two sides remains a singleton. If at least one agent has a positive endowment of money, then the above discussion tells us that the impossibility of Theorem 1 persists. The only remaining corner to escape the impossibility is when $|N| = 3$ and no agent has a positive endowment of money. Unfortunately, the positive news from the case of $|M| = 1$ or $|W| = 1$ with no money does not extend.

Theorem 2 *For non-bipartite problems without money, if $|N| = 3$ and $\mathcal{U} \supseteq \mathcal{U}^{slin}$, then no rule for $\mathcal{U}$ is strategy-proof, ex post efficient, and ex ante individually rational.*

We prove Theorem 2 in Appendix A. Since it is specific to the non-bipartite case it neither implies nor is implied by Theorem 1.

If we further drop the assumption that matches are pairwise, then we admit coalition formation problems. Pairwise and (non-)bipartite matching problems can be modeled as coalition formation problems where agents derive negative utility from being a member of a coalition with two or more other agents. Our negative results extend straightforwardly to the coalition formation problem with three or more agents.

Ordinal rules in the absence of money When $\omega = 0$, an interesting restriction that we may impose on a rule is that it be invariant to all information other than how agents rank their possible partners. A rule φ with domain $\mathcal{U} \subseteq \mathcal{U}^{slin}$ is *ordinal* if it only depends on induced preferences over partners. That is, for every pair $u, u' \in \mathcal{U}$, if for each $i \in N$, $R(u_i) = R(u'_i)$, then $\varphi(u) = \varphi(u')$. Ordinality is particularly relevant when the model has as a primitive each agent's preferences over partners. For each $m \in M$, let $\mathcal{P}_m$ be the set of linear orders over $W \cup \{m\}$. Define, for each $w \in W$, $\mathcal{P}_w$ similarly. For each $i \in N$ and each $\succeq_i \in \mathcal{P}_i$, let $\mathcal{U}(\succeq_i) \subseteq \mathcal{U}_i^{slin}$ be the set of utility functions that induce $\succeq_i$. If the starting point is a profile of preferences, we may describe a problem by $\succeq \in \times_{i \in N} \mathcal{P}_i \equiv \mathcal{P}$. To consider ordinal rules as defined above is to consider *preference-based* rules, that is, rules that map $\mathcal{P}$ to Π.

A preference-based rule φ is *deterministic* if for each $\succeq \in \mathcal{P}$, $\varphi(\succeq) = \sigma$ such that $\sigma \in \Sigma$. Alcalde and Barberà (1994) show that no preference-based and deterministic rule is strategy-proof, *ex post* Pareto-efficient and *ex post* individually rational. Does dropping the restriction that the rule be deterministic allow an escape from this incompatibility? We define particular extensions of strategy-proofness and individual rationality to non-deterministic rules and explain how Theorem 1 answers this question in the negative.

For the preference-based setting, a natural notion of strategy-proofness of non-deterministic rules is *stochastic-dominance (SD) strategy-proofness*. It says that any change that an agent can effect by misreporting his preferences leads to a worsening in the sense of stochastic dominance. Equivalently, a preference-based rule φ is SD strategy-proof if for each $\succeq \in \mathcal{P}$, each $i \in N$, each $\succeq'_i \in \mathcal{P}_i$, and each $u_i \in \mathcal{U}(\succeq_i)$,

$$u_i(\varphi_i(\succeq)) \geq u_i(\varphi_i(\succeq'_i, \succeq_{-i})).$$

If a preference-based rule is SD strategy-proof, then it naturally defines a strategy-proof and ordinal rule for $\mathcal{U}^{slin}$.

A similar definition of *SD individual rationality* says that each agent receives a lottery that stochastically dominates being alone with certainty. However, this is equivalent to ex post individual rationality and therefore implies ex ante individual rationality.

Notice that preference-based rules are a subset of the rules we have considered above, since they uniquely correspond to ordinal rules for $\mathcal{U}^{slin}$. We thus have the following corollary to Theorem 1, which extends the result of Alcalde and Barberà (1994) to non-deterministic matching.

Corollary 1 *For bipartite problems, if $|M| \geq 2$ and $|W| \geq 2$, then no preference-based rule is SD strategy-proof, ex post Pareto-efficient, and SD individually rational.*

Sönmez (1999) shows that for non-bipartite problems, no preference-based and deterministic rule is strategy-proof, *ex post* Pareto-efficient and *ex post* individually rational. We extend this result to non-deterministic rules in the following corollary to Theorems 1 and 2.

Corollary 2 *For non-bipartite problems, if $|N| \geq 3$, then no preference-based rule is SD strategy-proof, ex post Pareto-efficient, and SD individually rational.*

Appendices

A Impossibility for non-bipartite matching with $|N| = 3$

Since we restrict attention to the case of three agents, we name them 1,2, and 3, and assume the potential partners of an agent include every other agent. We show that strategy-proofness, ex post Pareto-efficiency, and ex ante individual rationality are incompatible for the case of $\omega = 0$ and $\mathcal{U} \supseteq \mathcal{U}^{slin}$.

Without the bipartite structure of the model, not every fractional matching can be interpreted as a probability distribution over deterministic matchings. That is, the Birkhoff-von Neumann theorem does not apply.

For this 3-agent model, Σ consists of the following four deterministic matchings:

$$\sigma^0 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma^{12} \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma^{13} \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \sigma^{23} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We therefore add another feasibility constraint requiring for every fractional matching π the existence of four numbers $p^0, p^{12}, p^{13}, p^{23} \in [0,1]$ such that $p^0 + p^{12} + p^{13} + p^{23} = 1$ and $\pi = p^0\sigma^0 + p^{12}\sigma^{12} + p^{13}\sigma^{13} + p^{23}\sigma^{23}$.

To prove the incompatibility of strategy-proofness, ex post Pareto-efficiency, and ex ante individual rationality, we consider first a property that is implied by their combination. Given $u \in \mathcal{U}^{slin}$, and a pair $l, m \in N$, say that *m is best for l at u* if, for each $n \in N$, $m\ R(u_l)\ n$. If m is best for l and l is best for m at u, say that *l and m are mutually best at u*. Say that φ *respects mutual bests* if, for each $u \in \mathcal{U}^{slin}$ such that there is a pair $l, m \in N$ that are mutually best, $\varphi(u) = \sigma^{lm}$. That is, whenever a pair of agents most prefer being matched together, then φ matches them fully.

Ex post Pareto-efficiency of π says that the weight on an element of Σ is zero if it is Pareto-dominated by another element. As a consequence, if π is ex post Pareto-efficient at $u \in \mathcal{U}^{slin}$, and if there is a pair of agents mutually best for each other at u, then π puts zero weight on σ^0.

Proposition 1 *Let $N = \{1,2,3\}$. If $\varphi : \mathcal{U}^{slin} \to \Pi$ is strategy-proof, ex post Pareto-efficient, and ex ante individually rational, then it respects mutual bests.*

Proof: Suppose that φ is strategy-proof, ex post Pareto-efficient, and ex ante individually rational.

For the remainder of the proof, we fix $u_3 \in \mathcal{U}_3^{slin}$ and prove that as long as 1 and 2 are mutually best, they are matched to one another. Consequently, we denote a profile of preferences only as a pair of preference relations, one for each of 1 and 2.

Let $u^0 \in \mathcal{U}^{slin}$ be such that

$$u_1^0(\pi_1) = \pi_{12} - \varepsilon\pi_{13}$$
$$\text{and}$$
$$u_2^0(\pi_2) = \pi_{21} - \varepsilon\pi_{23}.$$

Let $\pi^{00} \equiv \varphi(u_1^0 u_2^0)$. Suppose that $\pi^{00} \neq \sigma^{12}$. Then by ex post Pareto-efficiency, $\pi_{13}^{00} > 0$ or $\pi_{23}^{00} > 0$. Without loss of generality, suppose that $\pi_{13}^{00} > 0$.

Let α be large enough that $\frac{\alpha}{\alpha+2} > \frac{\frac{\pi_{12}^{00}+\varepsilon}{1+\varepsilon}+\varepsilon}{1+\varepsilon}$ and let $u^1 \in \mathcal{U}^{slin}$ be such that

$$u_1^1(\pi_1) = \pi_{12} - \alpha\pi_{13}$$
$$\text{and}$$
$$u_2^1(\pi_2) = \pi_{21} - \alpha\pi_{23}.$$

Since $\pi_{12}^{00} < 1$, $\frac{\pi_{12}^{00}+\varepsilon}{1+\varepsilon} < 1$. Thus, $\frac{\frac{\pi_{12}^{00}+\varepsilon}{1+\varepsilon}+\varepsilon}{1+\varepsilon} < 1$. So if α is sufficiently large, this condition is satisfied.

Let $\pi^{10} \equiv \varphi(u_1^1, u_2^0), \pi^{01} \equiv \varphi(u_1^0, u_2^1)$, and $\pi^{11} \equiv \varphi(u_1^1, u_2^1)$.

Since φ is strategy-proof, 1 does not benefit by misreporting u_1^1 at the preference profile u^0, so

$$\begin{array}{rrcl} & u_1^0(\pi_1^{10}) & \leq & u_1^0(\pi_1^{00}) \\ \Rightarrow & \pi_{12}^{10} - \varepsilon\pi_{13}^{10} & \leq & \pi_{12}^{00} - \varepsilon\pi_{13}^{00} \\ \Rightarrow & \pi_{12}^{10} - \varepsilon(1-\pi_{12}^{10}) & \leq & \pi_{12}^{00}. \end{array}$$

Thus, $\pi_{12}^{10} \leq \frac{\pi_{12}^{00}+\varepsilon}{1+\varepsilon}$. By definition of α, $\frac{\alpha}{\alpha+2} > \frac{\frac{\pi_{12}^{00}+\varepsilon}{1+\varepsilon}+\varepsilon}{1+\varepsilon}$. Combining this with $\pi_{12}^{10} \leq \frac{\pi_{12}^{00}+\varepsilon}{1+\varepsilon}$,

$$\frac{\alpha}{\alpha+2} > \frac{\pi_{12}^{10}+\varepsilon}{1+\varepsilon}. \tag{7}$$

Since φ is individually rational, $u_1^1(\pi_1^{11}) \geq 0$ and $u_2^1(\pi_2^{11}) > 0$. Together, these imply

$$\pi_{12}^{11} \geq \frac{\alpha}{\alpha+2}. \tag{8}$$

From (7) and (8),

$$\begin{array}{rrcl} & \pi_{12}^{11} & > & \frac{\pi_{12}^{10}+\varepsilon}{1+\varepsilon} \\ \Rightarrow & \pi_{12}^{11}(1+\varepsilon) & > & \pi_{12}^{10}+\varepsilon \\ \Rightarrow & \pi_{12}^{11} - \varepsilon(1-\pi_{12}^{11}) & > & \pi_{12}^{10} \\ \Rightarrow & \pi_{12}^{11} - \varepsilon\pi_{23}^{11}) & > & \pi_{12}^{10} - \varepsilon\pi_{23}^{10} \\ \Rightarrow & u_2^0(\pi_2^{11}) & > & u_2^0(\pi_2^{10}). \end{array}$$

However, this contradicts the strategy-proofness of φ since 2 benefits from misreporting u_2^1 at the preference profile (u_1^1, u_2^0). Thus, $\pi^{00} = \sigma^{12}$.

Now consider any $u_1 \in \mathcal{U}_1^{slin}$ such that 2 is best for 1. If $\varphi(u_1, u_2^0) \neq \sigma^{12}$, then 1 gains by misreporting u^0 at the preference profile (u_1, u_2^0). Finally, if $u \in \mathcal{U}^{slin}$ is such that 1 and 2 are mutually best, but $\varphi(u) \neq \sigma^{12}$, then 2 gains by misreporting u^0 at the preference profile u. Thus, φ respects mutual bests. $\square$

We now show that strategy-proofness is incompatible with respecting mutual bests.

Proposition 2 *Let $N = \{1,2,3\}$. If a rule $\varphi : \mathcal{U}^{slin} \to \Pi$ respects mutual bests, then it is not strategy-proof.*

Proof: Suppose that φ respects mutual bests. Fix γ and ε such that $\gamma > \varepsilon > 0$ and let

$u, u' \in \mathcal{U}^{slin}$ be such that

$$\begin{aligned} u_1(\pi_1) &= (\varepsilon+\gamma)\pi_{12} + \gamma\pi_{13}, \\ u_1'(\pi_1) &= \gamma\pi_{12} + (\varepsilon+\gamma)\pi_{13}, \\ u_2(\pi_2) &= (\varepsilon+\gamma)\pi_{23} + \gamma\pi_{21}, \\ u_2'(\pi_2) &= \gamma\pi_{23} + (\varepsilon+\gamma)\pi_{21}, \\ u_3(\pi_3) &= (\varepsilon+\gamma)\pi_{31} + \gamma\pi_{32}, \text{ and} \\ u_3'(\pi_3) &= \gamma\pi_{31} + (\varepsilon+\gamma)\pi_{32}, \end{aligned}$$

Let $\pi \equiv \varphi(u)$. Since φ respects mutual bests, $\varphi(u_1', u_{-1}) = \sigma^{13}$, $\varphi(u_2', u_{-2}) = \sigma^{12}$, and $\varphi(u_3', u_{-3}) = \sigma^{23}$. If φ is strategy-proof,

$$\begin{aligned} u_1(\pi_1) \geq u_1(\sigma_1^{13}) &\Rightarrow (\varepsilon+\gamma)\pi_{12} + \gamma\pi_{13} \geq \gamma \\ u_2(\pi_2) \geq u_1(\sigma_2^{12}) &\Rightarrow \gamma\pi_{21} + (\varepsilon+\gamma)\pi_{23} \geq \gamma \\ u_3(\pi_3) \geq u_3(\sigma_3^{23}) &\Rightarrow (\varepsilon+\gamma)\pi_{31} + \gamma\pi_{32} \geq \gamma \end{aligned}$$

Since, for each pair $i, j \in N$, $\pi_{ij} = \pi_{ji}$, summing these inequalities,

$$(\varepsilon + 2\gamma)(\pi_{12} + \pi_{13} + \pi_{23}) \geq 3\gamma.$$

However, feasibility requires that $\pi_{ij} + \pi_{ik} + \pi_{jk} \leq 1$. Therefore,

$$\frac{3\gamma}{\varepsilon + 2\gamma} \leq 1.$$

This inequality holds for all positive ε and γ. However, if $0 < \varepsilon < \gamma$, then $\frac{3\gamma}{\varepsilon+2\gamma} > 1$, a contradiction. □

Theorem 2 follows from the two propositions above.

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