Strategy-Proof Exchange under Trichotomous Preferences

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Abstract

We study the exchange of indivisible objects without monetary transfers when each agent may be endowed with and consume more than one object. We assume that each agent has *trichotomous* preferences in the sense that she

(A) partitions objects into desirable and undesirable ones,

(B) considers a bundle unacceptable if it contains undesirable objects that she was not already endowed with, and

(C) ranks acceptable bundles by their numbers of desirable objects.

On this domain, we show that there is an individually rational, Pareto-efficient, and strategy-proof mechanism that is also computationally efficient.

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1 Introduction

We study the problem of reallocating indivisible objects without monetary transfers. Unlike much of the earlier work on this problem,[1] we consider situations where each agent may be

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[1]See the literature following Shapley and Scarf [1974].

endowed with and consume more than one object. We require exchange to be balanced in the sense that agents have to end up with exactly the same number of objects as they were initially endowed with. Agents' preferences over individual objects are rather coarse: An object is either desirable or undesirable. Each agent considers a bundle unacceptable if it contains an undesirable objects that the agent was not already endowed with. Finally, agents rank acceptable bundles by the numbers of desirable objects that they obtain. Our contribution is to define an individually rational, Pareto-efficient, and strategy-proof mechanism that is computationally efficient. Our positive results are in marked contrast to the impossibility results that permeate the literature on multi-object exchange without monetary transfers [Sönmez, 1999, Biró et al., 2018].

Our interest in these exchange problems stems from our search for a solution to the problem of shift-reallocation. Millions of people in many different professions, from physicians to retail workers, engage in shift work. Shift plans are often made months in advance and scenarios like the following are common: A medical practice consists of four specialist consultants Drs A, B, C, and D. This practice is responsible for ensuring that emergency medical services in their specialty are available to a given hospital at all times. That is, each week, one of the four doctors is to be designated as being *on call*. Being on call is not desirable for these doctors. However, it is necessary for their practice to maintain privileges at the hospital. Since it is a chore that they must perform, in the interest of fairness, they agree to share the weeks equally so that each member of the group is on call every fourth week.[2] To facilitate planning, the call schedule is made six months at a time, taking the doctors' preferences into consideration. For instance, the schedule from January 1 to June 30 is announced in December, and is created on the basis of the doctors' preferences as of December. Thus, on January 1, each of the four doctors is responsible for their assigned weeks until June 30. While this initial assignment may be Pareto-efficient with regards to the doctors' December preferences, six months is a long time. At some point in the future, say the beginning of March, there may be scope for re-optimization based on current preferences. It may well happen that Dr. A would like to attend a conference the week of April 5, Dr. D would like to help at a clinic in a remote area on June 16, and Dr. C would like to go on vacation on May 23. If each of these doctors is obliged to be on call for the respective week, a three way trade could improve welfare in regards to current (as of March) preferences. We are interested in the design of mechanisms to identify such trades optimally and provide agents with incentives to reveal their private information about scheduling conflicts truthfully.[3]

[2] In the case of medical residents, such a restriction would not only be for reasons of fairness, but also for training purposes.

[3] Note that the re-optimization at the beginning of March in the above example is a static problem. We do not consider the issue of optimally timed matching as in Akbarpour et al. [2018].

The novel aspect of the model that we study is how we restrict agents' preferences. First, we assume that agents partition the set of all objects into desirable and undesirable ones. Second, no agent is ever willing to accept, i.e. would rather stick to her endowment, a bundle that contains one or more undesirable objects that the agent was not already endowed with. Finally, agents rank acceptable bundles by the number of desirable objects they obtain. We call preferences that satisfy these three assumptions *trichotomous*. In the context of shift exchange, trichotomous preferences can be interpreted as follows: a shift is undesirable (desirable) to a worker, if it is (not) in conflict with any other activities (e.g. paid work for other firms, further training/education, vacation) that the worker would like to or has already committed to. Furthermore, a worker is never willing to participate in an exchange that creates new scheduling conflicts in the sense that she is assigned new undesirable shifts. Put differently, workers will always find a way to fulfil the obligations associated with the initial shift schedule, while they face some hard constraints that limit which alternative shifts they can take on.

We focus on situations in which endowments are commonly known and exchange has to be balanced in the sense that each agent has to end up with the same number of goods as she was initially endowed with. In the context of shift exchange, these assumptions mean that all participants know the initial schedule of shifts and that a worker's total workload is fixed at the number of shifts in the initial schedule. Given our restriction to trichotomous preferences, the crucial piece of information that we need to elicit from an agent is which objects she finds desirable. An exchange mechanism asks each agent to reveal her set of desirable objects and then computes the final allocation of shifts on the basis of agents' reports as well as their commonly known endowments. We are interested in mechanisms that satisfy the following three key desiderata: *individual rationality* (no agent is assigned an unacceptable bundle), *Pareto-efficiency* (no further reallocation can make any agent better off without harming another), and *strategy-proofness* (no agent can ever profit from lying about her set of desirable objects). The main contribution of this paper is to define a new class of balanced exchange mechanisms satisfying all of these desiderata.[4] Our *Individually Rational Priority* (IRP) mechanism relies on a fixed ordering of the agents, independent of their reports, and picks a final allocation of goods in accordance with the following sequential process:

- In the first step, find the maximum number of desirable objects that the first agent can receive in an individually rational allocation. Call this number the "promise" to the first agent.

[4]For mechanisms that solve the unit endowment problem under general weak preferences, see Jaramillo and Manjunath [2012], Alcalde-Unzu and Molis [2011], Aziz and De Keijzer [2012], and Saban and Sethuraman [2013].

- In the t^{th} step, find the maximum number of desirable objects that the t^{th} agent can receive in an individually rational allocation subject to the constraints imposed by the promises made to the first $t-1$ agents. Call this number the "promise" to the t^{th} agent.

An IRP mechanism selects an outcome that meets the promises made to all of the agents in this process. We show that, given an ordering, an IRP mechanism is individually rational, Pareto-efficient, and strategy-proof. While individual rationality and Pareto-efficiency are almost by definition, it is significantly harder to show that this mechanism makes truthful revelation a weakly dominant strategy for the agents. The key feature of an IRP mechanism that makes establishing strategy-proofness difficult is that the report of an agent alters the set of individually rational allocations and can thereby affect the outcomes for all agents.[5] An agent who is not first in line may therefore hope that she can influence the promises to her predecessors in such a way that the mechanism promises her more truly desirable objects. In our proof, we develop an elaborate combinatorial argument to show that such hopes are misguided. Apart from their desirable allocative and incentive properties, we also show that IRP mechanisms are computationally efficient. More precisely, taking agents' desirable sets and a priority order over the agents as inputs, an IRP allocation may be computed in $O(mn^3)$ time, where m is the total number of objects and n is the number of agents.

Related Literature

Without rather strong restrictions on preferences, individual rationality, Pareto-efficiency, and strategy-proofness are incompatible when agents are endowed with and demand multiple objects [Sönmez, 1999, Biró et al., 2018]. Even other strategic properties such as immunity to manipulation via altering of endowments is generally incompatible with individual rationality and Pareto-efficiency [Klaus et al., 2006, Atlamaz and Klaus, 2007]. If we weaken the strategic requirement to say that successful manipulations are computationally intractable to compute, as suggested by Pini et al. [2011], then there are adaptations of the top-trading-cycles algorithm that satisfy this property while maintaining individual rationality and Pareto-efficiency Fujita et al. [2015], Sikdar et al. [2017, 2018], Phan and Purcell [2018].

The compatibility of individual rationality, Pareto-efficiency, and strategy-proofness in our model is driven by the restrictions that we impose on agents' preferences. For the two-sided one-to-one matching problem, the analog of our restriction—that each agent divides the set partners into desirable and undesirable ones and is indifferent among those in the same

[5] In a simple (sequential) priority mechanism, where agents take turns in picking their most preferred bundles without being subjected to individual rationality constraints, an agent's report can only influence the outcomes for agents who get to pick after her. It is precisely this feature which makes it straightforward to show that simple priority mechanisms are strategy-proof.

group—results in these properties being compatible as well [Bogomolnaia and Moulin, 2004]. In the context of balanced exchange in which agents are endowed with and demand multiple objects, Andersson et al. [2018] have independently considered a model similar to ours. By imposing stronger restrictions on agents' preferences—that all of the objects that a given agent is endowed with are identical—they are able to design individually rational and strategy-proof mechanisms that are not only Pareto-efficient, but also maximal in the number of objects that are traded. Since there is a trade-off between generality of the model and how demanding an objective can be satisfied, our results are logically independent from theirs.

Biró et al. [2018] consider a model where, as in Andersson et al. [2018], agents are endowed with copies of objects. While they also restrict attention to balanced exchange, they study preferences that are responsive to strict, as opposed to trichotomous, orderings over the objects. They study different variations of the top-trading-cycles algorithm to analyze trade-offs, since in their model Pareto-efficiency and strategy-proofness cannot be reconciled when individual rationality is required.

Organization of the paper

In Section 2 we introduce the model. In Section 3 we define theIRP algorithm and characterize the outcomes that this algorithm produces. In Section 4 we show that the direct mechanism induced by the IRP algorithm is strategy-proof. In Section 5 we show how to compute IRP matchings in polynomial time. In Section 6 we conclude and consider several extensions of the model, some of which are easily accommodated and others that lead to impossibility results. Proofs of the main results are in the Appendix. The Online Appendix contains proofs for several auxiliary results and for the additional results in Section 6.

2 The Model

Let $I = \{1, \ldots, n\}$ be a set of n agents and O be a finite set of objects. Each agent $i \in I$ is endowed with a set of objects $\Omega_i \subseteq O$. We assume throughout that $\Omega_i \cap \Omega_j = \emptyset$ whenever $i \neq j$ and that $O = \cup_{i \in I} \Omega_i$.[6] Each agent $i \in I$ has a weak preference relation $\succsim_i$ over 2^O.

A *matching* is a mapping $\mu : I \to 2^O$ that satisfies the following two requirements:

1. For any pair of distinct agents $i, j \in I$, $\mu(i) \cap \mu(j) = \emptyset$.

2. For any agent $i \in I$, $|\mu(i)| = |\Omega_i|$.

[6] In Section 6, we describe how to handle cases where some objects are not in the endowment of any agent, i.e. cases where $\cup_{i \in I} \Omega_i \subsetneq O$.

We require exchange to be *balanced* in the sense that each agent must end up with as many objects as she brought to the market. Let $\mathcal{M}$ be the set of all matchings. Given a profile of preferences $\succsim \equiv (\succsim_i)_{i \in I}$ and a matching $\mu \in \mathcal{M}$, μ is *individually rational under* $\succsim$ if each agent finds her assignment to be at least as desirable as her endowment—that is, for each $i \in I$, $\mu(i) \succsim \Omega_i$. The matching that assigns each agent her endowment is individually rational. A set of objects $O' \subseteq O$ such that $\Omega_i \succ_i O'$ is *unacceptable* and a set of objects $O'' \subseteq O$ such that $O'' \succeq_i \Omega_i$ is *acceptable* to agent $i \in I$. Next, $\mu' \in \mathcal{M}$ *Pareto-improves* $\mu \in \mathcal{M}$ if, for each $i \in I$, $\mu'(i) \succsim_i \mu(i)$ and, for some $i \in I$, $\mu'(i) \succ_i \mu(i)$. If there is no μ' that Pareto-improves μ, then μ is *Pareto-efficient.*

In the following, we develop assumptions about agents' preferences that play a key role in the remainder of this paper. Throughout, we fix an agent $i \in I$, i's endowment Ω_i, and a set of *desirable* objects $A_i \subseteq O$ for i. Our first assumption is that, when attention is restricted to acceptable sets of objects, more desirable objects are always better.[7]

Assumption 1 (More is better)**.** For any two sets $B_i, C_i \in 2^O$ such that $B_i \succsim_i \Omega_i$ and $C_i \succsim_i \Omega_i$, $B_i \succ_i C_i$ if and only if $|B_i \cap A_i| > |C_i \cap A_i|$.

Assumption 1 implies that i is indifferent between any two acceptable bundles that contain the same number of desirable objects. Next, we describe two different assumptions about which sets of objects an agent finds acceptable. We show below that both assumptions are equivalent when we restrict attention to balanced exchange. Our first assumption describes a scenario in which agent i exhibits an *aversion to unendowed undesirable objects* in the sense that i is not willing to accept a set of objects that contains one or more undesirable objects that she was not endowed with, no matter how many desirable objects the set contains.

Assumption 2 (Aversion to unendowed undesirable objects)**.** For any $B_i \in 2^O$, $B_i \succsim_i \Omega_i$ if and only if $B_i \subseteq A_i \cup \Omega_i$.

In the context of shift exchange, Assumption 2 means that a worker is never willing to create additional scheduling conflicts by accepting a shift for which she already knows that she will not be available. By contrast, any shift schedule that leaves i with only desirable shifts and undesirable endowed shifts is deemed acceptable.

Our next assumption describes a scenario in which agent i exhibits an aversion to unfavourable terms of trade in the sense that i requires at least one new desirable object for each of her endowed objects.

[7] In principle, the assumptions that follow allow agents to find it desirable to end up with strictly more objects than they brought to the market. Since we restrict attention to balanced exchange and assume that agents' endowments are known to all, it does not matter for our analysis how an agent ranks bundles with greater cardinality than her endowment.

Assumption 3 (Aversion to unfavourable terms of trade)**.** For any $B_i \in 2^O$, $B_i \succsim_i \Omega_i$ if and only if $|(B_i \cap A_i) \setminus \Omega_i| \geq |\Omega_i \setminus B_i|$.

In the context of shift exchange, Assumption 3 requires that a worker is willing to give up one of her endowed shifts if and only if she gets one new shift for which she has no scheduling conflicts. The assumption is sensible if, for example, i is contractually obliged to work at least $|\Omega_i|$ shifts or needs to work (and be compensated for) at least $|\Omega_i|$ shifts to pay her bills. We now establish that Assumptions 2 and 3 produce exactly the same set of individually rational matchings.

Proposition 1. *Let $\Omega_i = (\Omega_i)_{i\in I}$ be a profile of endowments and let $A = (A_i)_{i\in I}$ be a profile of desirable sets. Assume that, for each $i \in I$, both $\succsim_i$ and $\succsim_i'$ satisfy Assumption 1, that $\succsim_i$ satisfies Assumption 2, and that $\succsim_i'$ satisfies Assumption 3. Let $\succsim = (\succsim_i)_{i\in I}$ and $\succsim' = (\succsim_i')_{i\in I}$.*

A matching $\mu \in \mathcal{M}$, is individually rational under either $\succsim$ or $\succsim'$ if and only if, for all $i \in I$, $|(\mu(i) \cap A_i) \setminus \Omega_i| = |\Omega_i \setminus \mu(i)|$. In particular, a matching is individually rational under $\succsim$ if and only if it is individually rational under $\succsim'$.

Proposition 1 tells us that if we restrict attention to balanced exchange, then whether we assume that preferences exhibit aversion to unendowed undesirable objects or aversion to unfavourable terms of trade, makes no difference. Hence, we summarize our assumptions on preferences by means of the following definition

Definition 1. Fix an agent $i \in I$, an endowment $\Omega_i \subseteq O$, and a desirable set $A_i \subseteq O$. We say that i's preferences over sets of objects $\succsim_i$ are *trichotomous* with respect to A_i and Ω_i if $\succsim_i$ satisfies Assumption 1 and either Assumption 2 or Assumption 3.

Before proceeding, we discuss the relationship of our trichotomous preference domain to other approaches in the related literature. For that purpose, note first that we allow for the case where $\Omega_i \cap A_i \notin \{\emptyset, \Omega_i\}$, i.e. the case where some of i's endowed objects are desirable, while others are undesirable. Hence, we need information about endowments *and* desirable sets of objects in order to describe agents' preferences. To highlight this feature of our model, we have opted for the name trichotomous, as opposed to *dichotomous* as in Bogomolnaia and Moulin [2004].[8] Next, note that we also allow for the possibility that a non-empty strict subset of j's endowment Ω_j is desirable for $i \neq j$. This is an important way that ours is different from the the independent contribution of Andersson et al. [2018].

[8] Bogomolnaia and Moulin [2004] consider a two-sided one-to-one matching market in which each agent partitions potential match partners from the other side of the market into acceptable, i.e. better than being left unmatched, and unacceptable, i.e. worse than being left unmatched, ones. Since each agent is endowed with and cannot consume more than one "object", agents' preferences are entirely described by their sets of acceptable partners.

There, agents are assumed to either like, or find desirable, none or all of the objects another agent brings to the market. While this assumption may be sensible in the context of time banks, which is the leading application in Andersson et al. [2018], it is hard to justify in the context of shift exchange. In the latter application, we can think of objects in i's desirable set A_i as shifts where i has no scheduling conflicts at all, of objects in $\Omega_i \setminus A_i$ as shifts that i is responsible for but would like to trade away to engage in other activities (e.g. additional paid labor for other firms, further training/education, vacation), and of objects in $O \setminus (A_i \cup \Omega_i)$ as shifts that i is not able to take on due to existing scheduling conflicts. Clearly, it is possible that i has no scheduling conflicts for one shift o that was assigned to some agent $j \neq i$, while she is not able to take on another shift $p \neq o$ that was also assigned to j.

Henceforth, we assume that the preferences of all agents are trichotomous. For each agent $i \in I$, the acceptable sets of objects and the ranking of these acceptable sets are then completely identified by her endowment Ω_i and her desirable set A_i. We also assume from now on that endowments are known and fixed at the profile $\Omega = (\Omega_i)_{i \in I}$. In the context of shift exchange, the assumption of known endowments is reasonable since one would expect firms and workers to know who is supposed to work when (if no shift exchanges were to take place) once an initial assignment of shifts has been determined.

Now, given that we assume preferences to be trichotomous and that we assume endowments to be fixed and known, the only additional information that we need in order to know how agents rank individually rational matchings is each agent's desirable set. We assume that this set is private information to be elicited from each agent. In particular, agent i reports an element of $\mathcal{A} \equiv 2^O$ that she claims to be her desirable set. A *mechanism* is a mapping $\varphi : \mathcal{A}^n \to \mathcal{M}$ that associates a matching to each profile of desirable sets. Given a profile $A \in \mathcal{A}^n$, we denote the set of objects that i receives under mechanism φ by $\varphi_i(A)$. A mechanism is *individually rational* if it selects an individually rational matching for every profile of desirable sets. Recall that our assumptions about preferences imply that μ is individually rational at $A \in \mathcal{A}$ if and only if $\mu(i) \subseteq A_i \cup \Omega_i$. Similarly, a mechanism is *Pareto-efficient* if it selects a Pareto-efficient matching for every profile of desirable sets. Finally, a mechanism is *strategy-proof* if no agent can ever benefit by lying about her desirable set. Our assumptions about preferences imply that an individually rational mechanism φ is strategy-proof if there are no $A \in \mathcal{A}^n$, $i \in I$, and $\hat{A}_i \in \mathcal{A}$ such that $\varphi_i(\hat{A}_i, A_{-i}) \subseteq A_i \cup \Omega_i$ and $|\varphi_i(\hat{A}_i, A_{-i}) \cap A_i| > |\varphi_i(A) \cap A_i|$. Our main contribution is to define a mechanism that is invidually rational, Pareto-efficient, and strategy-proof on the domain of trichotomous preferences.

3 The Individually Rational Priority Algorithm

In this section, we introduce an algorithm that produces individually rational and Pareto-efficient matchings. Like sequential priority, agents take turns picking their most preferred sets of objects according to some exogenous priority ranking. The crucial difference with vanilla sequential priority is that our algorithm constrains each agent to choose in a way that is compatible with individual rationality for all agents. Since the set of individually rational matchings varies in response to each agent's report, an agent can influence the choice set of any other agent—not just those with lower priority. This feature of our algorithm makes it much harder to show that the induced direct mechanism is strategy-proof, which is what we turn to in the next section, than to show that a simple sequential priority mechanism has this property.

We now describe our algorithm. In the following, we equate an agent's index $i \in I = \{1, \dots, n\}$ with her priority. Note that lower indices correspond to higher priority. For the remainder of this section, fix a profile of desirable sets $A \in \mathcal{A}$. The *individually rational priority (IRP) algorithm* for A proceeds as follows:

Step 0: Let $\mathcal{M}^0(A)$ be the set of all individually rational matchings at A.

Step $\boldsymbol{t \in \{1, \dots, n\}}$: Let

$$K^t(A) \equiv \max_{\mu \in \mathcal{M}^{t-1}(A)} |\mu(t) \cap A_t|$$

be the *promise to agent* t and

$$\mathcal{M}^t(A) = \{\mu \in \mathcal{M}^{t-1}(A) : |\mu(t) \cap A_t| = K^t(A)\}$$

be the set of all matchings in $\mathcal{M}^{t-1}(A)$ that comply with the promise to t.

The above definitions imply that for any pair t, t' such that $t' \leq t$ and any $\mu \in \mathcal{M}^t(A)$, $|\mu(t') \cap A_{t'}| = K^{t'}(A)$. In particular, all matchings in $\mathcal{M}^n(A)$ are welfare equivalent for all agents. We call any $\mu \in \mathcal{M}^n(A)$ an *IRP outcome* for A.

We now establish that all IRP outcomes are individually rational and Pareto-efficient. First, for any $t \in \{1, \dots, n\}$, we have $\mathcal{M}^{t-1}(A) \subseteq \mathcal{M}^t(A)$. Hence, $\mathcal{M}^n(A) \subseteq \mathcal{M}^0(A)$, which implies that IRP outcomes are individually rational. Second, let $\mu \in \mathcal{M}^n(A)$. The construction of the sequence $\{\mathcal{M}^t(A)\}_{t=1}^n$ implies that there is no matching in $\mathcal{M}^0(A)$ that Pareto-dominates μ. For any matching $\nu \in \mathcal{M} \setminus \mathcal{M}^0(A)$, there is at least one agent $j \in I$ for whom individual rationality is violated, i.e. $\mu(j) \succsim_j \Omega_j \succ_j \nu(j)$. Hence, there is no matching in $\mathcal{M}$ that Pareto-dominates μ. The following result summarizes the findings of this paragraph.

Theorem 1. *Any $\mu \in \mathcal{M}^n(A)$ is individually rational and Pareto-efficient.*

In the remainder of this section, we develop a characterization of matchings in the sequence $\{\mathcal{M}^t(A)\}_{t=1}^n$ that plays a crucial role for the remainder of our analysis. For this purpose, we introduce some additional definitions.

Given a matching $\mu \in \mathcal{M}^0(A)$, a *cycle of* μ is a sequence $C = (i^1, o^1, \ldots, i^M, o^M)$ of M distinct agents and M distinct objects such that, for each $m \in \{1, \ldots, M\}$, $o^{m-1} \in \mu(i^m)$ and $o^m \notin \mu(i^m)$, where $o^0 \equiv o^M$. For the following discussion, fix an arbitrary cycle $C = (i^1, o^1, \ldots, i^M, o^M)$ of μ. For any $m \in \{1, \ldots, M\}$, we say that i^m *points to* o^m and that o^m *points to* i^{m+1}, where $i^{M+1} \equiv i^1$. Thus, C is a sequence of agent-object pairs such that each agent i^m points to an object o^m that she does not get at μ, i.e. an object $o^m \notin \mu(i^m)$, and each object o^m points to the agent who gets it at μ, i.e. the agent i^{m+1} for whom $o^m \in \mu(i^{m+1})$. Let $I(C) \equiv \{i^1, \ldots, i^M\}$ be the set of agents involved in C and let $O(C) \equiv \{o^1, \ldots, o^M\}$ be the set of objects involved in C. Let $\mu + C$ denote the matching that results from μ by executing the trades designated by C, i.e., for each $k \in I$,

$$(\mu + C)(k) \equiv \begin{cases} \mu(k) & \text{if } k \notin I(C) \\ (\mu(k) \setminus \{o^{m-1} : i^m = k\}) \cup \{o^m : i^m = k\} & \text{if } k \in I(C). \end{cases}$$

Motivated by the definition of $\mu + C$, we say that, for any $m \in \{1, \ldots, M\}$, i^m *trades* o^{m-1} *for* o^m *in* C. We say that C

- is *individually rational at* A if, for each $m \in \{1, \ldots, M\}$, $o^m \in \Omega_{i^m} \cup A_{i^m}$,
- *increases (decreases) i's welfare* if $|(\mu + C)(i) \cap A_i| > (<) \ |\mu(i) \cap A_i|$, and
- *does not affect i* if it neither increases nor decreases i's welfare of i.

Finally, we say that C is a *constrained improvement cycle (CIC) of μ for t at A* if C

1. increases t's welfare, and
2. does not affect any agent $t' < t$.

With these definitions, we now state the second result of this section.

Theorem 2. *For any $t \in \{1, \ldots, n\}$, we have that*

(i) $\mu \in \mathcal{M}^t(A)$ only if there does not exist a CIC of μ for some $t' \leq t$ at A, and

(ii) if $\mu \in \mathcal{M}^{t-1}(A)$ and there does not exist a CIC of μ for t at A, then $\mu \in \mathcal{M}^{t}(A)$.

To gain some intuition for Theorem 2, consider the t^{th} step of the IRP algorithm. Here, the number of desirable objects awarded to agent t is determined subject to the constraints imposed by the promises to agents $t' < t$. These constraints are expressed in the restriction to matchings in $\mathcal{M}^{t-1}(A)$. Now fix an arbitrary matching $\mu \in \mathcal{M}^{t}(A)$. If C is a CIC of μ for $t' \leq t$ at A, then $\mu + C \in \mathcal{M}^{t'-1}(A)$ given that $\mu \in \mathcal{M}^{t}(A) \subseteq \mathcal{M}^{t'-1}(A)$. Since C increases the welfare of t', we find that $\mu \notin \mathcal{M}^{t'}(A)$ and hence, given that $t' \leq t$, $\mu \notin \mathcal{M}^{t}(A)$. For the proof of part (ii), pick an arbitrary $\nu \in \mathcal{M}^{t-1}(A)$ that leaves t strictly better off than μ. Then there must exist an object o that is desirable for t and that t gets in ν but not in μ. In our proof, we use o as well as the other assignments in μ and ν to construct a CIC of μ for j at A.

4 Individually Rational Priority and Strategy-proofness

In this section, we show that a direct mechanism induced by the IRP algorithm for any given priority order is strategy-proof. Mechanism φ is an *individually rational priority (IRP) mechanism* if $\varphi(A) \in \mathcal{M}^{n}(A)$ for all $A \in \mathcal{A}$. Recall that, since we have fixed the order over agents, any IRP mechanism is guaranteed to induce the same welfare for all agents. Recall also that $\{\mathcal{M}^{t}(A)\}_{t=1}^{n}$ summarizes the progression of the IRP algorithm for input A. .

For the remainder of this section, fix a true profile of desirable sets $A \in \mathcal{A}$, an agent $i \in I$, and a possible manipulation for i $\hat{A}_i \in \mathcal{A}_i$. Denote the profile $(\hat{A}_i, A_{-i})$ by $\hat{A}$. We establish that an IRP mechanism is strategy-proof by showing that, when the true profile of desirable sets is A, i is no better off at $\hat{A}$ than he is at A.

We start by showing that i does not benefit by falsely claiming that an object is desirable when it is not. The first of two lemmas considers the case where she falsely claims as desirable an object o that she is not endowed with. It says that the the number of trades that she receives can only change if she receives o at every matching produced by the IRP algorithm at $\hat{A}$.

Lemma 1. *If, for some $o \notin \Omega_i \cup A_i$, $\hat{A}_i = A_i \cup \{o\}$ and $K^i(\hat{A}) \neq K^i(A)$, then, for each $\mu \in \mathcal{M}^{n}(\hat{A})$, $o \in \mu(i)$.*

By our assumption that preferences are trichotomous, Lemma 1 implies that falsely claiming a single unendowed undesirable object to be desirable is not beneficial. The next lemma shows that an agent cannot benefit from falsely claiming that one of her endowed objects is desirable.

Lemma 2. *If, for some $o \in \Omega_i \setminus A_i$, $\hat{A}_i = A_i \cup \{o\}$, then for each $\mu \in \mathcal{M}^{i}(\hat{A})$, $K^i(A) \geq |\mu(i) \cap A_i|$.*

Lemmas 1 and 2 imply that expanding the set of desirable objects can never be profitable for an agent. Next, we consider the case where i falsely declares a desirable object to be undesirable.

Lemma 3. *If, for some $o \in A_i$, $\hat{A}_i = A_i \setminus \{o\}$, then $K^i(A) \geq K^i(\hat{A})$.*

Since Lemma 3 is the main step in proving strategy-proofness of IRP mechanisms, we provide a sketch of its proof. Note first that when i shrinks her desirable set from A_i to $\hat{A}_i = A_i \setminus \{o\}$, she places an additional constraint on the choices of all agents in the IRP algorithm since she can no longer receive o. Put differently, the set of individually rational matchings shrinks from $\mathcal{M}^0(A)$ to $\mathcal{M}^0(\hat{A}) \subseteq \mathcal{M}^0(A)$. It is then immediate that a contraction can never be profitable for the first agent in line since such a manipulation would only shrink her own choice set. Things are potentially different for $i > 1$: she might hope that shrinking the choice sets of agents $j < i$ forces these agents to leave her with a choice set $\mathcal{M}^{i-1}(\hat{A})$ that contains some matching μ that she prefers to all matchings in $\mathcal{M}^{i-1}(A)$. So assume that there exists such a matching $\mu \in \mathcal{M}^{i-1}(\hat{A})$. Then $|\mu(i) \cap A_i| > K^i(A)$. Since $\mu \notin \mathcal{M}^{i-1}(A)$ but $\mu \in \mathcal{M}^0(A)$, there exists a smallest integer $j < i$ such that $\mu \notin \mathcal{M}^j(A)$. By Theorem 2, there exists a CIC C of μ for j at A. Given that the only difference between A and $\hat{A}$ is that i ranks o as undesirable in $\hat{A}$, i must point to o in C. Since $o \in A_i \setminus \Omega_i$, C cannot decrease the welfare of i. If $\mu + C \in \mathcal{M}^{i-1}(A)$, we obtain a contradiction to our assumption that $|\mu(i) \cap A_i| > K^i(A)$ since there is a matching that i could have picked in the IRP algorithm at A that is at least as good as μ. If $\mu + C \notin \mathcal{M}^{i-1}(A)$, Theorem 2 again implies that there has to be a CIC C' of $\mu + C$ at A for some $j' < i$. The main part of our proof of Lemma 3 shows that we can construct a third CIC C^* from C and C' such that $\mu + C^*$ (A) makes every agent $k \leq \min\{j, j'\}$ at least as well off as $(\mu + C) + C'$, and (B) makes i weakly better off than μ. Iterating this argument, we eventually obtain a contradiction to our assumption that $|\mu(i) \cap A_i| > K^i(A)$ since there must exist at least one matching in $\mathcal{M}^{i-1}(A)$ that makes i weakly better off than μ.

Finally, we appeal to Lemmas 1, 2, and 3 to prove that any IRP mechanism is strategy-proof.

Theorem 3. *Any IRP mechanism is strategy-proof.*

Before proceeding, we point out that our restriction to trichotomous preferences is important for strategy-proofness of IRP mechanisms. For example, even for the case of singleton endowments, a sequential priority over the set of individually rational matchings is not generally strategy-proof if agents' preferences have more than three indifference classes.

5 Computing Individually Rational Priority Matchings

In this section, we define a polynomial time algorithm (Algorithm 1) to compute matchings in $\mathcal{M}^n(A)$ based on network flows. The procedure described in Section 3 relies on an enumeration of all of the individually rational matchings ($\mathcal{M}^0(A)$), the subset of these matchings in which 1 receives the most desirable objects ($\mathcal{M}^1(A)$), and so on. The problem is that sizes of these sets are an exponential function of the number of agents and objects. The algorithm that we now develop uses the fact that each of these sets is the set of solutions to a particular network flow problem.

For the following discussion, fix an endowment Ω and a profile of desirable sets A. We work with a directed graph (V, E). The set of vertices V contains three vertices corresponding to each $i \in I$, call them i, i^A, and i^U. It also contains one vertex corresponding to each $o \in O$. Finally, it contains a source S and a sink T. That is,

$$V \equiv \left[\bigcup_{i\in I}\{i, i^A, i^U\}\right] \cup O \cup \{S, T\}$$

The set of edges E contains an edge from S to each $i \in I$, from each $i \in I$ to i^A and i^U, from i^A to each object in A_i, from i^U to each object in $\Omega_i \setminus A_i$, and from each $o \in O$ to T. That is,

$$\begin{aligned} E \quad \equiv \quad & \{(S,i) : i \in I\} \\ \cup \quad & \{(i, i^A), (i, i^U) : i \in I\} \\ \cup \quad & \{(i^A, o) : o \in A_i\} \\ \cup \quad & \{(i^U, o) : o \in \Omega_i \setminus A_i\} \\ \cup \quad & \{(o, T) : o \in O\}. \end{aligned}$$

Roughly speaking, the idea here is that copy i^A of agent i collects all desirable objects, while copy i^U collects all undesirable endowments. Throughout our algorithm, we only vary the capacities of each edge. The capacity of edge $e \in E$ is $q(e)$ and we initialize q as follows:

$$\begin{aligned} q(S,i) &= |\Omega_i| && i \in I \\ q(i,i^A) &= |\Omega_i| && i \in I \\ q(i,i^U) &= |\Omega_i \setminus A_i| && i \in I \\ q(i^A,o) &= 1 && i \in I, o \in A_i \\ q(i^U,o) &= 1 && i \in I, o \in \Omega_i \\ q(o,T) &= 1 && o \in O. \end{aligned}$$

We say that $f : E \to \mathbb{N}$ is an (integer) flow, if $f(e) \leq q(e)$, for all $e \in E$, and

$\sum_{e\in\delta^-(v)} f(e) = \sum_{e\in\delta^+(v)} f(e)$, for all $v \in V \setminus \{S,T\}$.[9] The value of a flow f through i is $f(S,i)$ and the value of a flow f is $v(f) = \sum_{i\in I} f(S,i)$. Since $q(S,i) = |\Omega_i|$, for all $i \in I$, the maximum flow through i cannot be larger than $|\Omega_i|$. Since for all $o \in O$, $i \in I$, and $D \in \{A,U\}$, $q(i^D, o) \in \{0,1\}$, a flow cannot use two edges involving the same object but different agents. If no flow g has higher value than f, then f is a *maximum flow*. Let $\textsc{MaxFlow}(q)$ be the value of a maximum flow from S to T when the capacities are q. By the above observation that the flow through i cannot exceed $|\Omega_i|$, for every q that we consider, $\textsc{MaxFlow}(q) \leq \sum_{i\in I} |\Omega_i|$ and, for the initial value of q, $\textsc{MaxFlow}(q) = \sum_{i\in I} |\Omega_i|$ since we can assign each agent her endowment at that capacity vector. Given $\mu \in \mathcal{M}$, we say that μ *corresponds to* f if, for each $i \in I$ and each $o \in \mu(i) \cap A_i$, $f(i^A, o) = 1$ and, for each $i \in I$ and each $o \in \mu(i) \cap \Omega_i \setminus A_i$, $f(i^u, o) = 1$.

Algorithm 1 iteratively minimizes each agent i's *undesirable object capacity*, $q(i, i^U)$, by solving a series of network flow problems.

Algorithm 1 Procedure to compute IRP

procedure $\mathrm{IRP}(V, E, q)$
  for $t = 1$ to $t = N$ **do** ▷ Loop invariant: $\textsc{MaxFlow}(q) = \sum_i |\Omega_i|$
    if $q(t, t^U) > 0$ **then** ▷ t has undesirable object capacity
      $\hat{q} = q$
      $\hat{q}(t, t^U) = \hat{q}(t, t^U) - 1$ ▷ Decrement t's undesirable capacity by 1
      while $\textsc{MaxFlow}(\hat{q}) = \sum_i |\Omega_i|$ **do** ▷ As long as the loop invariant holds
        $q = \hat{q}$
        $\hat{q}(t, t^U) = \hat{q}(t, t^U) - 1$ ▷ Decrement t's undesirable capacity by 1
  return q

The following result summarizes the key properties of Algorithm 1.

Theorem 4. *Let $A \in \mathcal{A}$ be a profile of desirable sets and let Ω be an endowment. Suppose that V, E, and q are defined based on A and Ω as described above.*

1. *A flow f is a maximum flow in $(V, E, \mathrm{IRP}(V, E, q))$ if and only if it corresponds to an IRP outcome.*

2. *Fixing Ω, the complexity of Algorithm 1 is $O(mn^3)$, where $m = |O|$.*

6 Conclusion and Discussion

We have considered a model of exchange of indivisible goods without monetary transfers and multi-unit demand and supply. The following corollary summarizes our main results.

[9]By $\delta^+(v)$ we denote the edges that originate at node v and by $\delta^-(v)$ the edges that point to v.

Corollary 1. *Fix an endowment Ω and assume that all agents have trichotomous preferences.*

1. *IRP mechanisms are individually rational, Pareto-efficient, and strategy-proof.*
2. *It is computationally efficient to find IRP outcomes.*

In light of Corollary 1, IRP mechanisms are promising solutions to balanced exchange problems in which agents have trichotomous preferences are endowed with more than one indivisible object. In the remainder of this section, we consider several extensions of our model. Our main result, the compatibility of individual rationality, Pareto-efficiency, and strategy-proofness, is robust to some but not all of these extensions. Unless explicitly mentioned otherwise, we maintain the assumption that exchange is balanced and that endowments are commonly known and fixed at some profile Ω. The proofs for all additional results from this section are in the Online Appendix.

6.1 Unendowed objects

We have so far assumed that every object is part of some agent's endowment, i.e. that $\cup_{i\in I}\Omega_i = O$. In this subsection, we show how to accommodate the case where some of the objects are "unendowed," i.e. $\cup_{i\in I}\Omega_i \subsetneq O$. Let $O^u = O \setminus (\cup_{i\in I}\Omega_i)$ be the set of unendowed objects and $r = |O^u|$ be the number of unendowed objects.

We first consider a variant of our model in which agents may find some unendowed objects desirable, but exchange is still required to be balanced with respect to agents' endowments and agents' preferences are still assumed to be trichotomous. This model is identical to our basic model described in Section 2 except that we now allow for the possibility that $O^u \neq \emptyset$. We refer to this variant of our model as *"balanced exchange with unendowed objects"*. In the context of shift exchange, this describes a situation in which each worker is pre-contracted to perform a fixed number of shifts and extra shifts are to be filled by additional contracts with outside workers. We now explain how to extend our IRP mechanisms to handle unendowed objects while maintaining individual rationality, Pareto-efficiency, and strategy-proofness. Since we assume that agents' preferences are trichotomous, recall that a matching problem can be summarized by a profile of desirable sets $A \in \mathcal{A}^n$. To handle the r objects that have to remain unassigned in balanced exchange, we introduce a "dummy agent" d who is endowed with the objects in O^u, finds every object desirable, and has trichotomous preferences. Formally, we move to a modified economy with agents $I' = I \cup \{d\}$, where $\Omega_d = O^u$ and $A_d = O$. It is straightforward that a matching is individually rational and Pareto-efficient for the problem (A, A_d) in the modified economy if and only if the induced matching for the non-dummy agents in I is individually rational and Pareto-efficient for the matching problem

A in the original economy. By Theorem 3, any IRP mechanism for the modified economy is strategy-proof. Hence, our results imply that there exists a mechanism that is individually rational, Pareto-efficient, and strategy-proof for balanced exchange with unendowed objects.

Next, we consider a variant of our model in which agents may end up with more (but not fewer) objects than they brought to the market. That is, a matching is a mapping $\mu : I \to 2^O$ that satisfies the following two requirements:

1. For any pair of distinct agents $i, j \in I$, $\mu(i) \cap \mu(j) = \emptyset$.

2. For any agent $i \in I$, $|\mu(i)| \geq |\Omega_i|$.

We still assume that agents' preferences are trichotomous. We refer to the this variant of our model as *"minimum guarantees with unendowed objects"*. This applies to situations in which it is commonly known that an agent does not mind receiving additional desirable objects. For example, when allocating timeshares for vacation properties, it is reasonable to assume that an owner prefers to receive more weeks than she was initially entitled to provided that she, or someone she knows, can use the additional weeks. To extend the IRP algorithm, we modify the definition of the base set $\mathcal{M}^0$ to take into account that agents may now pick more objects than they brought to the market as long as no agent is made worse off than keeping her endowed objects. In the Online Appendix, we use a similar construction to the one for balanced exchange with unendowed objects to show that our results imply that any IRP mechanism is strategy-proof. We obtain the following.

Proposition 2. *For minimum guarantees with unendowed objects, IRP mechanisms are individually rational, Pareto-efficient, and strategy-proof.*

Finally, we consider the implications of adding to the definition of a matching the requirement that every object be assigned to some agent. This requirement is not, in general compatible, with individual rationality. Consider, for instance, a matching problem where each agent $i \in I$ only finds her endowment to be desirable, i.e. $A_i = \Omega_i$. Then assigning to any agent an object that she was not endowed with violates individual rationality. Thus, there is no individually rational matching that assigns every object to some agent when there are unendowed objects.

6.2 Other Preference Domains

We discuss here two alternative restrictions on the preference domain. First, we consider a "minimal" departure from the domain of trichotomous preferences that allows each agent to have one "special" desirable object that she prefers over all other desirable objects. We do so while maintaining the requirement that agents to rank bundles by their numbers of desirable objects. Second, we consider the possibility of complementarities in agents' preferences over sets of objects.

Strict preference for one object The assumption that preferences are trichotomous requires each agent to be indifferent between all sets that contain the same number of desirable objects. As a minimal departure from the trichotomous domain, suppose that agents can possibly distinguish between sets that contain no more than one desirable trade, but only if the comparison involves one particular object. Formally, consider an agent $i \in I$ who has a desirable set $A_i \subseteq O$. Designate for i a special object $a_i \in A_i$. We weaken Assumption 1 only with regards to comparing sets containing a single desirable object, when one of them contains a_i.

Assumption 4. For any two sets $B_i, C_i \in 2^O$ such that $B_i \succsim_i \Omega_i$ and $C_i \succsim_i \Omega_i$, $B_i \succ_i C_i$ if and only if either

1. $|B_i \cap A_i| > |C_i \cap A_i|$, or
2. $|B_i \cap A_i| = |C_i \cap A_i| = 1$ and $B_i \cap A_i = \{a_i\}$.[10]

Restricted to sets that either do not contain a_i or contain more than one desirable object Assumptions 1 and 4 are equivalent. We now define a minimal departure from trichotomous preferences:

Definition 2. Fix an agent $i \in I$, an endowment $\Omega_i \subseteq O$, and a desirable set $A_i \subseteq O$. We say that i's preferences over sets of objects $\succsim_i$ are *minimally non-trichotomous* with respect to A_i and Ω_i if there is $a_i \in A_i$ such that $\succsim_i$ satisfies Assumption 4 and either Assumption 2 or Assumption 3

This is enough of a weakening in the preference restriction to lead to an incompatibility of individual rationality, Pareto-efficiency, and strategy-proofness. Intuitively, this means that whenever agents may treat desirable objects as anything other than perfect substitutes, these three requirements are incompatible.

Proposition 3. *Suppose that there are at least three agents, at least one of whom is endowed with more than one object. If agents' preferences are either trichotomous or minimally non-trichotomous, then individual rationality, Pareto-efficiency, and strategy-proofness are incompatible.*

[10]Since we require B_i and C_i to be acceptable to agent i, this case is possible only when $|A_i \cap \Omega_i| \leq 1$.

Note that Proposition 3 is not a "maximal domain" result. It does not say that adding to the domain of trichotomous preferences a single preference relation that is not trichotomous leads us to an impossibility result. Rather, Proposition 3 tells us that, when all minimally non-trichotomous preferences are allowed, there is no mechanism that satisfies our three desiderata.

Complementarities The assumption of trichotomous preferences requires agents to view all desirable objects as "perfect substitutes". While substitutability might be a reasonable assumption in shift exchange, complementarities play an important role in other potential applications of our model. For example, following Ergin et al. [2017], a recent literature has started investigating the design of organ exchanges for applications in which patients may require more than one donor. If patients need multiple donors in order to be treated successfully, it is reasonable to expect that they (and their incompatible direct donors) are only willing to engage in exchange if they end up with exactly the required number of compatible donors. Here, we consider alternatives to our definition of trichotomous preferences that allow agents to view objects as complements.

For the following discussion, fix an agent $i \in I$, a desirable set $A_i \subseteq O$, and some number $k_i \in \mathbb{N}_+$. We start by considering the case of perfect complements, which is likely to be the most relevant case for applications to multi-donor organ exchange. We modify Assumption 2 to account for the possibility that there is some *target* $k_i \geq 1$ such that agent i is only willing to participate in trading if she ends up with exactly k_i desirable objects.

Assumption 5. For any $B_i \in 2^O$, $B_i \succsim_i \Omega_i$ if and only if $B \subseteq A_i \cup \Omega_i$ and either $B_i \cap A_i = k_i$ or $B_i = \Omega_i$.

Based on this modification of Assumption 2, we define another domain of preferences.

Definition 3. Fix an agent $i \in I$, an endowment $\Omega_i \subseteq O$, a desirable set $A_i \subseteq O$, and a target $k_i \in \mathbb{N}_+$. We say that i's preferences over sets of objects $\succsim_i$ are *all-or-nothing* with respect to A_i, Ω_i, and k_i if $\succsim_i$ satisfies Assumption 1 and Assumption 5.

If i's preferences are all-or-nothing, then she has at most three indifference classes: bundles that give her exactly k_i desirable objects, her endowment, and all other bundles. Also, if $k_i > 1$, it is immediate from the definitions that if $\succsim_i$ is all-or-nothing with respect to A_i, Ω_i, and k_i, then it is not trichotomous. Finally, since we assume agents' endowments to be fixed, the information to be elicited from an agent i with all-or-nothing preferences is the pair (A_i, k_i).We now develop a simple adaptation of the IRP algorithm that maintains the desirable properties IRP mechanisms have on the trichotomous domain. Given a profile of desirable sets A and a profile of targets k, the algorithm works as follows:

Step 0: Let $\mathcal{M}^0(A, k)$ be the set of all individually rational matchings at the all-or-nothing preferences with respect to (A, k).

Step $\boldsymbol{t \in \{1, \ldots, n\}}$: Let

$$K^t(A, k) \equiv \begin{cases} k_i & \text{if there is } \mu \in \mathcal{M}^{t-1}(A, k) \text{ such that } |\mu(i) \cap A_i| = k_i \\ |\Omega_i \cap A_i| & \text{otherwise} \end{cases}$$

and

$$\mathcal{M}^t(A, k) = \left\{ \mu \in \mathcal{M}^{t-1}(A, k) : \begin{array}{ll} |\mu(t) \cap A_t| = K^t(A, k) & \text{if } K^t(A, k) = k_t \text{ and} \\ \mu(t) = \Omega_t & \text{otherwise} \end{array} \right\}.$$

As is the case when objects are perfect substitutes, an IRP mechanism selects an allocation in $\mathcal{M}^n(A, k)$. Though all-or-nothing preferences are distinct from trichotomous preferences, IRP mechanisms have the same desirable properties on this domain of preferences as well.

Proposition 4. *For all-or-nothing preferences, any IRP mechanism is individually rational, Pareto-efficient, and strategy-proof.*

Next, we consider the case where each agent has a lower bound rather than a target for the number of desirable objects that she receives. For example, it might be that there is some (unmodelled) fixed cost of participating in exchange that keeps agents from participating when they get too few desirable objects. Given an agent $i \in I$, desirable set $A_i \subseteq O$, and lower bound $\underline{k}_i \in \mathbb{N}_+$, the following modification of Assumption 5 describes such preferences

Assumption 6. For any $B_i \in 2^O$, $B_i \succsim_i \Omega_i$ if and only if $B_i \subseteq A_i \cup \Omega_i$ and either $B_i \cap A_i \geq k_i$ or $B_i = \Omega_i$.

Based on this modification of Assumption 5, we define another domain of preferences.

Definition 4. Fix an agent $i \in I$, an endowment $\Omega_i \subseteq O$, a desirable set $A_i \subseteq O$, and a lower bound $\underline{k}_i \in \mathbb{N}_+$. We say that i's preferences over sets of objects $\succsim_i$ are *lower bound constrained* with respect to $(A_i, \Omega_i, \underline{k}_i)$ if $\succsim_i$ satisfies Assumption 1 and Assumption 6.

As for the case of all-or-nothing preferences, the information that needs to be elicited from an agent i with lower bound constrained preferences is the pair $(A_i, \underline{k}_i)$. Unfortunately, for lower bound constrained preferences, our three requirements are incompatible.

Proposition 5. *If agents' preferences are lower bound constrained, then individual rationality, Pareto-efficiency, and strategy-proofness are incompatible.*

Intuitively, this means that whenever agents treat the objects as complements, but are willing to trade off between them to some extent, no desirable mechanisms exist.

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Appendices

A Proof of Proposition 1

Assume first that matching μ is individually rational under $\succsim$ and fix an agent $i \in I$. Matching μ is individually rational for i under $\succsim$ if and only if $|\mu(i)| = |\Omega_i|$ and $\mu(i) \subseteq \Omega_i \cup A_i$. The last two relations are equivalent to $|\Omega_i \setminus \mu(i)| = |\mu(i) \setminus \Omega_i|$ and $\mu(i) \setminus \Omega_i \subseteq A_i$, which are in turn equivalent to $|(\mu(i) \cap A_i) \setminus \Omega_i| = |\Omega_i \setminus \mu(i)|$ and $|\mu(i)| = |\Omega_i|$.

Next, assume that μ is individually rational under $\succsim'$ and fix an agent $i \in I$. Matching μ is individually rational for i under $\succsim'$ if and only if $|\mu(i)| = |\Omega_i|$ and $|(\mu(i) \cap A_i) \setminus \Omega_i| \geq |\Omega_i \setminus \mu(i)|$. Since

$$|\mu(i)| \geq |(\mu(i) \cap A_i) \setminus \Omega_i| + |\mu(i) \cap \Omega_i|$$

and

$$|\Omega_i| = |\Omega_i \setminus \mu(i)| + |\mu(i) \cap \Omega_i|,$$

$|\mu(i)| = |\Omega_i|$ and $|(\mu(i) \cap A_i) \setminus \Omega_i| \geq |\Omega_i \setminus \mu(i)|$ are compatible with each other if and only if $|(\mu(i) \cap A_i) \setminus \Omega_i| = |\Omega_i \setminus \mu(i)|$ and $|\mu(i)| \geq |(\mu(i) \cap A_i) \setminus \Omega_i| + |\mu(i) \cap \Omega_i|$.

B Proof of Theorem 2

We first introduce a notion of generalized CICs where we allow agents, but not objects, to reoccur. A *generalized cycle of* μ is a sequence $C \equiv (i^1, o^1, \ldots, i^M, o^M)$ of M (not necessarily distinct) agents and M distinct objects such that, for each $m \in \{1, \ldots, M\}$, $o^{m-1} \in \mu(i^m)$ and $o^m \notin \mu(i^m)$ (where $o^0 \equiv o^M$). For the following discussion, fix an arbitrary generalized cycle $C = (i^1, o^1, \ldots, i^M, o^M)$ of μ. As for the case of cycles, we denote the matching that results by implementing the exchanges that C specifies by $\mu + C$, i.e., for each $k \in I$, we set

$$(\mu + C)(k) \equiv \begin{cases} \mu(k) & \text{if } k \notin I(C) \\ (\mu(k) \setminus \{o^{m-1} : i^m = k\}) \cup \{o^m : i^m = k\} & \text{if } k \in I(C) \end{cases}$$

We say that C is an *individually rational generalized cycle*, if, for all m, $o^m \in \Omega_{i^m} \cup A_{i^m}$. Furthermore, we refer to i^1 as the *head* of C. A *generalized constrained improvement cycle (GCIC) of* μ *for* j *at* A is an individually rational generalized cycle $C = (i^1, o^1, \ldots, i^M, o^M)$ of μ at A that satisfies the following additional conditions:

1. j is the head of C: $i^1 = j$,
2. C improves its head's welfare: $o^M \in \Omega_j \setminus A_j$, $o^1 \in A_j$, and, for any $m \neq 1$ such that $i^m = j$ and $o^m \in \Omega_j \setminus A_j$, we have that $o^{m-1} \in \Omega_j \setminus A_j$
3. C does not affect agents with higher priority than its head: for any m such that $i^m < j$, either $\{o^{m-1}, o^m\} \subseteq A_{i^m}$ or $\{o^{m-1}, o^m\} \subseteq \Omega_{i^m} \setminus A_{i^m}$.

We now start our proof by verifying that if C is an individually rational generalized cycle, then $\mu + C$ is individually rational as well.

Lemma 4. *Let A and $\mu \in \mathcal{M}^0(A)$. If C is an individually rational generalized cycle of μ at A, then $\mu + C \in \mathcal{M}^0(A)$.*

Proof. Let $C = (i^1, o^1, \ldots, i^M, o^M)$ and $\nu \equiv \mu + C$. First, for each $o \in O$, since the objects in C are distinct o cannot be in both $\nu(k)$ and $\nu(l)$ for distinct agents k and l. Next, for each $k \in I$, $\nu(k) \subseteq \Omega_k \cup A_k$ since $\mu(k) \subseteq \Omega_k \cup A_k$ and $\nu(k) \setminus \mu(k) \subseteq \Omega_k \cup A_k$ given that C is individually rational. Finally, we verify that for each $k \in I$, $|\nu(k)| = |\Omega_k|$. Since the objects in C are distinct and since $o^{m-1} \in \mu(i^m)$ as well as $o^m \notin \mu(i^m)$ for all m, we have

$$\begin{aligned}
|\nu(k)| &= |\mu(k) \setminus \{o^{m-1} : i^m = k\}| + |\{o^m : i^m = k\}| \\
&= |\mu(k)| - |\{o^{m-1} : i^m = k\}| + |\{o^m : i^m = k\}| \\
&= |\Omega_k|
\end{aligned}$$

□

For the remainder of the proof, we fix $t \in \{1, \ldots, n\}$. We first argue that the absence of GCICs is necessary for a matching to be compatible with the guarantee for t at A.

Lemma 5. *If $\mu \in \mathcal{M}^t(A)$, then there is no GCIC of μ for some $t' \leq t$ at A.*

Proof. Suppose to the contrary that $C \equiv (i^1, o^1, \ldots, i^M, o^M)$ is a GCIC of μ for some $t' \leq t$ at A. Let $\nu \equiv \mu + C$. Since C does not affect any agent $t'' < t'$, $|\nu(t'') \cap A_{t''}| = |\mu(t'') \cap A_{t''}| = K^{t''}(A)$ for all $t'' < t'$. By Lemma 4, ν is individually rational. Thus, combining the last two observations with the fact that $\mu \in \mathcal{M}^t(A) \subseteq \mathcal{M}^{t'-1}(A)$, $\nu \in \mathcal{M}^{t'-1}(A)$. However, since C increases t''s welfare, we have $|\nu(t') \cap A_{t'}| > |\mu(t') \cap A_{t'}| = K^{t'}(A)$ and thus $\mu \notin \mathcal{M}^{t'-1}(A)$. Hence, we obtain a contradiction to $\mu \in \mathcal{M}^t(A)$. □

Since any CIC is, trivially, also a GCIC, Lemma 5 already shows the necessity part, part (i), of Theorem 2.

Showing that the absence of CICs is sufficient, i.e. part (ii) of Theorem 2, requires more work. We begin with an auxiliary lemma showing that while an agent $j < t$ may appear more than once in a GCIC of μ, it has to be the case that j either only trades desirable for desirable objects, or only trades undesirable endowed objects for undesirable endowed objects.

Lemma 6. *Let $j \in I$ be such that $j < t$, $\mu \in \mathcal{M}^{t-1}(A)$, and $C = (i^1, o^1, \ldots, i^M, o^M)$ be a GCIC of μ for t at A. If m and m' are such that $i^m = i^{m'} = j$, then either*

$$\{o^{m-1}, o^m, o^{m'-1}, o^{m'}\} \subseteq \Omega_j \setminus A_j$$

or

$$\{o^{m-1}, o^m, o^{m'-1}, o^{m'}\} \subseteq A_j.$$

Proof. Fix m and m' such that $i^m = i^{m'} = j$ and assume, without loss of generality, that $m < m' \leq M$. Since C does not affect any agent with higher priority than t, it suffices to show that either $\{o^m, o^{m'}\} \subseteq \Omega_j \setminus A_j$ or $\{o^m, o^{m'}\} \subseteq A_j$. Suppose, for the sake of contradiction, that neither is true. There are two cases. If $o^m \in \Omega_j \setminus A_j$ and $o^{m'} \in A_j$, consider

$$C' \equiv (i^1, o^1, \dots i^{m-1}, o^{m-1}, i^{m'}, o^{m'}, \dots, i^M, o^M).$$

Since C is a generalized individually rational cycle of μ, C' is a generalized individually rational cycle of μ as well. Since C is a GCIC for $t > j$ and $o^m \in \Omega_j \setminus A_j$, we have that $o^{m-1} \in \Omega_j \setminus A_j$ as well. Furthermore, since C is a GCIC for $t > j$, we also obtain that, for any $\tilde{m} \in \{1, \dots, m-1, m'+1, \dots, M\}$ such that $i^{\tilde{m}} \leq t-1$, $\{o^{\tilde{m}-1}, o^{\tilde{m}}\} \subseteq A_{i^{\tilde{m}}}$ or $\{o^{\tilde{m}-1}, o^{\tilde{m}}\} \subseteq \Omega_{i^{\tilde{m}}} \setminus A_{i^{\tilde{m}}}$ (where $o^0 = o^M$). Hence, subject to relabeling the agents, C' is a GCIC of μ for j at A, which we have already argued to be impossible. On the other hand, if $o^m \in A_j$ and $o^{m'} \in \Omega_j \setminus A_j$, the following sequence yields a similar contradiction: $(i^m, o^m, \dots, i^{m'-1}, o^{m'-1})$. Thus, either $\{o^m, o^{m'}\} \subseteq A_j$ or $\{o^m, o^{m'}\} \subseteq \Omega_j \setminus A_j$. □

Building on Lemma 6, our next auxiliary lemma states that it is sufficient to consider CICs since any recurrence of an agent can be eliminated to yield a shorter cycle.

Lemma 7. *Let $\mu \in \mathcal{M}^{t-1}(A)$. If there exists a GCIC of μ for t at A, then there exists an CIC of μ for t at A.*

Proof. Let $C \equiv (i^1, o^1, \dots, i^M, o^M)$ be a GCIC of μ for t at A. We show how to construct a CIC C' of μ for t on basis of C.

We first eliminate duplicates of any $j \in I(C)$ such that $t < j$. Suppose there are two indices m and m' such that $m < m'$ and $i^m = i^{m'} = j$. Then, since $\{o^{m-1}, o^m, o^{m'-1}, o^{m'}\} \subseteq \Omega_j \cup A_j$, the sequence $(i^1, o^1, \dots, i^{m-1}, o^{m-1}, i^{m'}, o^{m'}, \dots, i^M, o^M)$ is also a GCIC of μ for t at A that has one fewer instances of j than C.

Next, we eliminate duplicates of t. Suppose there is m such that $m > 1$ and $i^m = t$. If $o^m \in \Omega_t \setminus A_t$, then since C improves its head's welfare, we must have $o^{m-1} \in \Omega_t \setminus A_t$, so that $(i^1, o^1, \dots, i^{m-1}, o^{m-1})$ is a GCIC of μ for t at A with one fewer instances of t than C. If $o^m \in A_t$, then $(i^m, o^m, \dots, i^M, o^M)$ is a GCIC of μ for t at A with one fewer instances of t.

Finally, we eliminate duplicates of $j \in I(C)$ such that $j < t$. Suppose there are two indices m and m' such that $m < m'$ and $i^m = i^{m'} = j$. By Lemma 6, we have that either $\{o^{m-1}, o^{m'}\} \subseteq \Omega_j \setminus A_j$ or $\{o^{m-1}, o^{m'}\} \subseteq A_j$. Hence, the sequence $(i^1, o^1, \dots, i^{m-1}, o^{m-1}, i^{m'}, o^{m'}, \dots, i^M, o^M)$ is a GCIC of μ for t at A that has one fewer instances of j than C. □

In the remainder of this proof, we establish that if $\mu \in \mathcal{M}^{t-1}(A)$ is such that $|\mu(t) \cap A_t| < K^t(A)$, then there is a GCIC of μ for t at A. By Lemma 7, we thereby establish the sufficiency part of Theorem 2.

Fix $\mu \in \mathcal{M}^{t-1}(A)$ such that $|\mu(t) \cap A_t| < K^t(A)$. We construct a GCIC of μ for t at A. We start with $i^1 = t$. Let $\nu \in \mathcal{M}^t(A)$. Since $|\mu(t) \cap A_t| < K^t(A) = |\nu(t) \cap A_t|$, there exists

$o^1 \in (\nu(t) \setminus \mu(t)) \cap A_t$. Suppose now that we have grown a sequence $(i^1, o^1, \ldots, i^M, o^M)$ of M agents and distinct objects in a way that satisfies the following four conditions:

$$\text{for each } m \geq 1, o^m \in \nu(i^m) \setminus \mu(i^m), \text{ and, for each } m \geq 2, o^{m-1} \in \mu(i^m) \setminus \nu(i^m), \tag{C1}$$

$$\text{for each } m \geq 1, o^m \in \Omega_{i^m} \cup A_{i^m}, \tag{C2}$$

$$\begin{gathered}\text{for each pair } m, m' \text{ such that } i^m = i^{m'} < t, \text{ either} \\ \{o^{m-1}, o^m, o^{m'-1}, o^{m'}\} \subseteq A_{i^m} \text{ or } \{o^{m-1}, o^m, o^{m'-1}, o^{m'}\} \subseteq \Omega_{i^m} \setminus A_{i^m}, \text{ and}\end{gathered} \tag{C3}$$

$$\text{for each } m \geq 2 \text{ such that } i^m = t, \{o^{m-1}, o^m\} \subseteq A_t. \tag{C4}$$

Clearly, (C1) to (C4) imply that if $o^M \in \mu(i^1) \cap (\Omega_{i^1} \setminus A_{i_1})$, then $(i^1, o^1, \ldots, i^M, o^M)$ is a GCIC of μ for $i^1 = t$ at A. We now show that if $o^M \notin \mu(i^1) \cap (\Omega_{i^1} \setminus A_{i^1})$, then we can find an agent-object pair (i^{M+1}, o^{M+1}) so that $(i^1, o^1, \ldots, i^{M+1}, o^{M+1})$ satisfies (C1) to (C4).

Note first that since ν and μ are both individually rational, we must have $\sum_{i \in I} |\nu(i)| = \sum_{i \in I} |\mu(i)| = \sum_{i \in I} |\Omega_i|$. Hence, $o^M \notin \mu(i^M)$ implies that there exists an agent $i^{M+1} \in I \setminus \{i^M\}$ such that $o^M \in \mu(i^{M+1})$. Since $o^M \in \nu(i^M)$, we have $o^M \in \mu(i^{M+1}) \setminus \nu(i^{M+1})$. We distinguish three cases according to the identity of i^{M+1}. In each of these cases, we identify an object o^{M+1} such that the sequence $(i^1, o^1, \ldots i^{M+1}, o^{M+1})$ satisfies the properties listed above.

Case 1: $\boldsymbol{i^{M+1} \in I \setminus \{i^1, \ldots, i^M\}}$ There are three subcases to consider:

Case 1.1: $\boldsymbol{i^{M+1} < t}$ and $\boldsymbol{o^M \in A_{i^{M+1}}}$ Since $\nu, \mu \in \mathcal{M}^{t-1}(A)$ and $i^{M+1} < t$, it follows that $|\nu(i^{M+1}) \cap A_{i^{M+1}}| = |\mu(i^{M+1}) \cap A_{i^{M+1}}|$. Since $o^M \in (\mu(i^{M+1}) \setminus \nu(i^{M+1})) \cap A_{i^{M+1}}$ there is $o^{M+1} \in (\nu(i^{M+1}) \setminus \mu(i^{M+1})) \cap A_{i^{M+1}}$. Furthermore, since $i^{M+1} \notin \{i^1, \ldots, i^M\}$ and, for each m, $o^m \in \nu(i^m)$, we have that $o^{M+1} \notin \{o^1, \ldots, o^M\}$.

Case 1.2: $\boldsymbol{i^{M+1} < t}$ and $\boldsymbol{o^M \in \Omega_{i^{M+1}} \setminus A_{i^{M+1}}}$ Since $\nu, \mu \in \mathcal{M}^{t-1}(A)$ and $i^{M+1} < t$, we must have $|\nu(i^{M+1}) \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})| = |\mu(i^{M+1}) \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})|$. Since $o^M \in (\mu(i^{M+1}) \setminus \nu(i^{M+1})) \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})$ there is $o^{M+1} \in (\nu(i^{M+1}) \setminus \mu(i^{M+1})) \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})$. Furthermore, since $i^{M+1} \notin \{i^1, \ldots, i^M\}$ and, for each m, $o^m \in \nu(i^m)$, we have that $o^{M+1} \notin \{o^1, \ldots, o^M\}$.

Case 1.3: $\boldsymbol{i^{M+1} > t}$ Since ν and μ are individually rational, we must have $\nu(i^{M+1}) \subseteq \Omega_{i^{M+1}} \cup A_{i^{M+1}}$, $\mu(i^{M+1}) \subseteq \Omega_{i^{M+1}} \cup A_{i^{M+1}}$, and $|\nu(i^{M+1})| = |\mu(i^{M+1})| = |\Omega_{i^{M+1}}|$. Given these facts and that $o^M \in \mu(i^{M+1}) \setminus \nu(i^{M+1})$, there is $o^{M+1} \in \nu(i^{M+1}) \setminus \mu(i^{M+1})$. Furthermore, since $i^{M+1} \notin \{i^1, \ldots, i^M\}$ and for each m, $o^m \in \nu(i^m)$, we have that $o^{M+1} \notin \{o^1, \ldots, o^M\}$.

Case 2: $\boldsymbol{i^{M+1} \in \{i^2, \ldots, i^M\} \setminus \{t\}}$ Let $m \leq M$ be such that $i^m = i^{M+1}$. We distinguish three subcases parallel to those of Case 1.

Case 2.1: $i^{M+1} < t$ and $o^M \in A_{i^{M+1}}$ We first show that $o^m \in A_{i^{M+1}}$. Otherwise, $o^m \in \Omega_{i^{M+1}} \setminus A_{i^{M+1}}$. Consider the following sequence:

$$C' \equiv (i^{M+1}, o^M, i^M, o^{M-1}, \dots, i^{m+1}, o^m).$$

Since we assume that $(i^1, o^1, \dots, i^M, o^M)$ satisfies (C1), we have that $o^{m'} \notin \nu(i^{m'+1})$ and $o^{m'} \in \nu(i^{m'})$ for all $m' \in \{1, \dots, M+1\}$. Hence, C' is a cycle of ν. We now show that C' is a GCIC of ν for i^{M+1} at A. By (C2) above, $o^{m'} \in \Omega_{i^{m'+1}} \cup A_{i^{m'+1}}$ for all $m' \in \{1, \dots, M\}$, so that C' is individually rational at A. Next, note that (C3) and $i^{M+1} < t$ imply that C' does not affect any agent with higher priority than i^{M+1}. Finally, (C3), $i^{M+1} < t$, and $o^M \in A_{i^{M+1}}$ imply the remaining requirements for C' to be a GCIC of ν for i^{M+1} at A. Since C' is a GCIC of ν for i^{M+1} at A, by Lemma 5, $\nu \notin \mathcal{M}^{i^{M+1}}(A)$. Since $i^{M+1} < t$, we must have $\mathcal{M}^t(A) \subseteq \mathcal{M}^{i^{M+1}}(A)$ and thus $\nu \notin \mathcal{M}^t(A)$, which is a contradiction. Thus, $o^m \in A_{i^{M+1}}$.

Since m was arbitrary, (C3) and the just established fact imply that, for each $m' \leq M$ such that $i^{m'} = i^{M+1}$, $\{o^{m'-1}, o^{m'}\} \subseteq A_{i^{M+1}}$. Since $\nu, \mu \in \mathcal{M}^{t-1}(A) \subseteq \mathcal{M}^{i^{M+1}}(A)$, we have $|\nu(i^{M+1}) \cap A_{i^{M+1}}| = |\mu(i^{M+1}) \cap A_{i^{M+1}}|$. Since $o^M \in (\mu(i^{M+1}) \setminus \nu(i^{M+1})) \cap A_{i^{M+1}}$ and, for each $m' \in \{2, \dots, M-1\}$, $o^{m'-1} \in \mu(i^{m'}) \setminus \nu(i^{m'})$ and $o^{m'} \in \nu(i^{m'}) \setminus \mu(i^{m'})$, we have that

$$\begin{gathered}|(\nu(i^{M+1}) \setminus \mu(i^{M+1})) \cap \{o^1, \dots, o^M\} \cap A_{i^{M+1}}| \\ < \\ |(\mu(i^{M+1}) \setminus \nu(i^{M+1})) \cap \{o^1, \dots, o^M\} \cap A_{i^{M+1}}|.\end{gathered}$$

Thus, combining the last two observations, there exists $o^{M+1} \in [(\nu(i^{M+1}) \setminus \mu(i^{M+1})) \setminus \{o^1, \dots, o^M\}] \cap A_{i^{M+1}}$.

Case 2.2: $i^{M+1} < t$ and $o^M \in \Omega_{i^{M+1}} \setminus A_{i^{M+1}}$ We first show that $o^m \in \Omega_{i^{M+1}} \setminus A_{i^{M+1}}$. Otherwise, $o^m \in A_{i^{M+1}}$. Consider the following sequence:

$$(i^m, o^m, \dots i^M, o^M)$$

Since $o^m \in A_{i^{M+1}}$ but $o^M \in \Omega_{i^{M+1}} \setminus A_{i^{M+1}}$, by the assumptions that we have made with regards to $(i^1, o^1, \dots, i^M, o^M)$, this sequence is a GCIC of μ for i^{M+1} at A. Hence, Lemma 5 implies that $\mu \notin \mathcal{M}^{i^{M+1}}(A)$. Since $\mu \in \mathcal{M}^t(A) \subseteq \mathcal{M}^{i^{M+1}}(A)$, we obtain a contradiction. Thus, $o^m \in \Omega_{i^{M+1}} \setminus A_{i^{M+1}}$.

Since m was arbitrary, (C3) and the just established fact imply that, for each $m' \leq M$ such that $i^{m'} = i^{M+1}$, $\{o^{m'-1}, o^{m'}\} \subseteq \Omega_{i^{M+1}}$. Since $\nu, \mu \in \mathcal{M}^{t-1}(A) \subseteq$

$\mathcal{M}^{i^{M+1}}(A)$, we obtain $|\nu(i^{M+1}) \cap A_{i^{M+1}}| = |\mu(i^{M+1}) \cap A_{i^{M+1}}|$ and hence $|\nu(i^{M+1}) \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})| = |\mu(i^{M+1}) \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})|$. Since $o^M \in (\mu(i^{M+1}) \setminus \nu(i^{M+1})) \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})$ and, for each $m' \in \{2, \ldots, M-1\}$, $o^{m'-1} \in \mu(i^{m'}) \setminus \nu(i^{m'})$ as well as $o^{m'} \in \nu(i^{m'}) \setminus \mu(i^{m'})$, we have that

$$\begin{gathered} |(\nu(i^{M+1}) \setminus \mu(i^{M+1})) \cap \{o^1, \ldots, o^M\} \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})| \\ < \\ |(\mu(i^{M+1}) \setminus \nu(i^{M+1})) \cap \{o^1, \ldots, o^M\} \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})|. \end{gathered}$$

Thus, there is $o^{M+1} \in [(\nu(i^{M+1}) \setminus \mu(i^{M+1})) \setminus \{o^1, \ldots, o^M\}] \cap (\Omega_{i^{M+1}} \setminus A_{i^{M+1}})$.

Case 2.3: $\boldsymbol{i^{M+1} > t}$ Since ν and μ are both feasible, $|\mu(i^{M+1}) \setminus \nu(i^{M+1})| = |\nu(i^{M+1}) \setminus \mu(i^{M+1})|$. However, since $o^M \in \mu(i^{M+1}) \setminus \nu(i^{M+1})$ and for each $m \in \{2, \ldots, M-1\}$, $o^{m-1} \in \mu(i^m) \setminus \nu(i^m)$ as well as $o^m \in \nu(o^m) \setminus \mu(o^m)$, we have that

$$|(\nu(i^{M+1}) \setminus \mu(i^{M+1})) \cap \{o^1, \ldots, o^M\}| < |(\mu(i^{M+1}) \setminus \nu(i^{M+1})) \cap \{o^1, \ldots, o^M\}|.$$

Thus, there is $o^{M+1} \in (\nu(i^{M+1}) \setminus \mu(i^{M+1})) \setminus \{o^1, \ldots, o^M\}$.

Case 3: $\boldsymbol{i^{M+1} = t}$ and $\boldsymbol{o^M \in A_{i^{M+1}}}$ By (C4), $|(\nu(t) \setminus \mu(t)) \cap \{o^1, \ldots, o^M\} \cap A_t| = |(\mu(t) \setminus \nu(t)) \cap \{o^1, \ldots, o^M\} \cap A_t|$. Since $|\nu(t) \cap A_t| > |\mu(t) \cap A_t|$, there is $o^{M+1} \in [(\nu(t) \setminus \mu(t)) \setminus \{o^1, \ldots, o^M\}] \cap A_t$.[11]

C Proofs from Section 4

Proof of Lemma 1. Let j be the highest priority agent for whom $K^j(\hat{A}) \neq K^j(A)$ and note that $K^i(A) \neq K^i(\hat{A})$ implies $j \leq i$.

We first show via induction on k that, for all $k \leq j$, $\mathcal{M}^{k-1}(A) \subseteq \mathcal{M}^{k-1}(\hat{A})$ and $o \in \mu(i)$ for each $\mu \in \mathcal{M}^{k-1}(\hat{A}) \setminus \mathcal{M}^{k-1}(A)$. For $k = 1$, both statements follow from $\hat{A}_i = A_i \cup \{o\}$ and $\hat{A}_l = A_l$ for all $l \neq i$. So consider some $k \in \{2, \ldots, j\}$ and assume that both statements have been shown for all $k' < k$. We have that

$$\begin{aligned} \mathcal{M}^{k-1}(A) &= \{\mu \in \mathcal{M}^{k-2}(A) : |\mu(k-1) \cap A_{k-1}| = K^{k-1}(A)\} \\ &\subseteq \{\mu \in \mathcal{M}^{k-2}(\hat{A}) : |\mu(k-1) \cap A_{k-1}| = K^{k-1}(A)\} \\ &= \{\mu \in \mathcal{M}^{k-2}(\hat{A}) : |\mu(k-1) \cap \hat{A}_{k-1}| = K^{k-1}(\hat{A})\} \\ &= \mathcal{M}^{k-1}(\hat{A}) \end{aligned}$$

[11] Recall that $i^{M+1} = t$ implies $o^M \in A_{i^{M+1}}$ (as otherwise there would have been no need to extend the sequence $i^1, o^1, \ldots, i^M, o^M$) so Cases 1, 2, and 3 are exhaustive.

Here, the subset relation follows from the inductive assumption that $\mathcal{M}^{k-2}(A) \subseteq \mathcal{M}^{k-2}(\hat{A})$ and the second equality follows from the definitions of j and $\hat{A}$ since $k-1 < j \leq i$. In order to show the second part of the statement for k, fix an arbitrary $\mu \in \mathcal{M}^{k-1}(\hat{A})$ such that $o \notin \mu(i)$. By construction of $M^{k-1}(\hat{A})$, we have $\mu \in \mathcal{M}^{k-2}(\hat{A})$. Since $o \notin \mu(i)$, the inductive assumption for $k-1$ implies $\mu \in \mathcal{M}^{k-2}(A)$. Since $k-1 < j \leq i$, we have that $|\mu(k-1) \cap A_{k-1}| = |\mu(k-1) \cap \hat{A}_{k-1}|$ and $K^{k-1}(A) = K^{k-1}(\hat{A})$. Combining the last two statements, we obtain $\mu \in \mathcal{M}^{k-1}(A)$.

We now use the just established statements for $k = j$ to complete the proof. Since $\mathcal{M}^{j-1}(A) \subseteq \mathcal{M}^{j-1}(\hat{A})$, $K^j(\hat{A}) \geq K^j(A)$ and the definition of j implies $K^j(\hat{A}) > K^j(A)$. Thus, $\mathcal{M}^j(\hat{A}) \subseteq \mathcal{M}^{j-1}(\hat{A}) \setminus \mathcal{M}^{j-1}(A)$. Hence, we must have $o \in \mu(i)$ for all $\mu \in \mathcal{M}^j(\hat{A})$. Since $j \leq i$, we have that $\mathcal{M}^i(\hat{A}) \subseteq \mathcal{M}^j(\hat{A})$. Combining the last two observations we obtain that $o \in \mu(i)$ for all $\mu \in \mathcal{M}^i(\hat{A})$, which proves the desired statement. □

Proof of Lemma 2. Since $\Omega_i \cup A_i = \Omega_i \cup \hat{A}_i$, $\mathcal{M}^0(\hat{A}) = \mathcal{M}^0(A)$. Since, for each $t < i$, $\hat{A}_t = A_t$, $\mathcal{M}^t(\hat{A}) = \mathcal{M}^t(A)$. Thus, $\mathcal{M}^{i-1}(\hat{A}) = \mathcal{M}^{i-1}(A)$. By definition of $K^i(A)$ and $\mathcal{M}^i(A)$, for each $\mu \in \mathcal{M}^i(A) \subseteq \mathcal{M}^{i-1}(\hat{A})$, $K^i(A) \geq |\mu(i) \cap A_i|$. □

Proof of Lemma 3. If $K^i(A) = K^i(\hat{A})$, there is nothing left to show. So for the remainder of the proof, assume that $K^i(A) \neq K^i(\hat{A})$. We argue that $K^i(A) > K^i(\hat{A})$.

Let j^* be the highest priority agent for whom $K^{j^*}(\hat{A}) \neq K^{j^*}(A)$. Note that $K^i(A) \neq K^i(\hat{A})$ implies $j^* \leq i$.

We argue first that $K^{j^*}(A) > K^{j^*}(\hat{A})$. Since $K^j(\hat{A}) = K^j(A)$ for all $j < j^*$ and $\hat{A}_k \subseteq A_k$ for all $k \in I$, we must have $\mathcal{M}^{j^*-1}(\hat{A}) \subseteq \mathcal{M}^{j^*-1}(A)$. The last subset relation implies $K^{j^*}(A) \geq K^{j^*}(\hat{A})$ and thus, given that we assumed $K^{j^*}(A) \neq K^{j^*}(\hat{A})$, we must have $K^{j^*}(A) > K^{j^*}(\hat{A})$.

Given that $K^{j^*}(A) > K^{j^*}(\hat{A})$, the statement of Lemma 3 follows if $j^* = i$. Henceforth, we assume that $j^* < i$. In our proof, we show that, for each $\mu \in \mathcal{M}^{i-1}(\hat{A})$, there is a CIC C^* of μ for j^* at A such that $\mu + C^* \in \mathcal{M}^{i-1}(A)$ and $|(\mu + C^*)(i) \cap A_i| \geq |\mu(i) \cap A_i|$. In particular, for any $\mu \in \mathcal{M}^{i-1}(\hat{A})$ there exists a $\mu' \in \mathcal{M}^{i-1}(A)$ such that $|\mu'(i) \cap A_i| \geq |\mu(i) \cap A_i|$. Hence, $K^i(A) \geq K^i(\hat{A})$ and thus, given that we have assumed $K^i(A) \neq K^i(\hat{A})$, $K^i(A) > K^i(\hat{A})$, which yields Lemma 3.

For the remainder of the proof of Lemma 3, we fix an arbitrary $\mu \in \mathcal{M}^{i-1}(\hat{A})$ and show that a CIC of μ for j^* at A with the desired properties exists. Since the proof is quite involved, we introduce 10 auxiliary claims, whose proofs are all relegated to an online appendix.

Our first auxiliary claim is that μ complies with the promise to j^*-1 in the IRP algorithm under A but fails to comply with the promise to j^*.

Claim 1. $\mu \in \mathcal{M}^{j^*-1}(A) \setminus \mathcal{M}^{j^*}(A)$

By Theorem 2, Claim 1 implies that there exists a CIC $C \equiv (i^1, o^1, \dots, i^M, o^M)$ of μ for j^* at A. The CIC C remains fixed throughout the remainder of the proof of Lemma 3. Our second auxiliary claim shows that i has to receive o in C.

Claim 2. $o \in (\mu + C)(i) \setminus \mu(i)$

Assume first that there does not exist a CIC of $\mu + C$ for any $j < i$ at A. By Claim 1 and Theorem 2, we have that $\mu + C \in \mathcal{M}^{i-1}(A)$. By Claim 2, we have that $o \in (\mu + C)(i) \setminus \mu(i)$ and thus, given that $o \in A_i$, $|(\mu + C)(i) \cap A_i| \geq |\mu(i) \cap A_i|$. Hence, C is a CIC with the desired properties and we are done.

Henceforth, we assume that there is an agent $\hat{j}^* < i$ and a CIC $\hat{C} \equiv (j^1, p^1, \dots, j^L, p^L)$ of $\mu + C$ for $\hat{j}^*$ at A. The CIC $\hat{C}$ remains fixed throughout the remainder of the proof of Lemma 3. We assume without loss of generality that there is no agent $j' < \hat{j}^*$ for whom there is a CIC of $\mu + C$ at A. Our next claim lists two important initial observations about $\hat{j}^*$, C, and $\hat{C}$.

Claim 3. $\hat{j}^* \geq j^*$ *and* $\{\mu + C, (\mu + C) + \hat{C}\} \subseteq \mathcal{M}^{\hat{j}^*-1}(A)$

In the following, we use C and $\hat{C}$ to show that there exists a third CIC $\tilde{C}$ of μ for j^* at A that satisfies the following properties:

$$\mu + \tilde{C} \in \mathcal{M}^{\hat{j}^*-1}(A) \tag{P1}$$

$$|(\mu + \tilde{C})(\hat{j}^*) \cap A_{\hat{j}^*}| \geq |((\mu + C) + \hat{C})(\hat{j}^*) \cap A_{\hat{j}^*}| \tag{P2}$$

$$|(\mu + \tilde{C})(i) \cap A_i| \geq |\mu(i) \cap A_i| \tag{P3}$$

Repeated application of this argument allows us to infer that there exists a CIC C^* of μ for j^* at A that weakly increases the welfare of i and that satisfies $\mu + C^* \in \mathcal{M}^{i-1}(A)$.[12] As mentioned above, the existence of such a CIC proves Lemma 3.

Before proceeding, we note that since $(\mu + C) + \hat{C} \in \mathcal{M}^{\hat{j}^*-1}(A)$, a CIC $\tilde{C}$ satisfies (P1) - (P3) if and only if

1. $\tilde{C}$ makes every agent $j' \leq \hat{j}^*$ at least as well off as $(\mu + C) + \hat{C}$, i.e. $|(\mu + \tilde{C})(j') \cap A_{j'}| \geq |((\mu + C) + \hat{C})(j') \cap A_{j'}|$ for all $j' \leq \hat{j}^*$, and

2. $\tilde{C}$ weakly increases the welfare of i, i.e. $|(\mu + \tilde{C})(i) \cap A_i| \geq |\mu(i) \cap A_i|$.

[12] To see this, note first that if $\mu + \tilde{C} \notin \mathcal{M}^{i-1}(A)$, then Theorem 2 and $\mu + \tilde{C} \in \mathcal{M}^{\hat{j}^*-1}(A)$ imply that there exists a CIC $\tilde{C}'$ of $\mu + \tilde{C}$ for some $j' \geq \hat{j}^*$. We can now apply our earlier finding to $\tilde{C}$ and $\tilde{C}'$ to get yet another CIC that satisfies the three properties listed above. Finally, note if $j' = \hat{j}^*$, then we have that $|((\mu + \tilde{C}) + \tilde{C}')(\hat{j}^*) \cap A_{\hat{j}^*}| > |\mu(\hat{j}^*) \cap A_{\hat{j}^*}|$ since $|(\mu + \tilde{C})(\hat{j}^*) \cap A_{\hat{j}^*}| \geq |((\mu + C) + \hat{C})(\hat{j}^*) \cap A_{\hat{j}^*}| \geq |\mu(\hat{j}^*) \cap A_{\hat{j}^*}|$ and since $\tilde{C}'$ is a CIC for $\hat{j}^*$. Since the set of agents is finite and since the maximum number of desirable objects an agent can obtain is finite, we must eventually find a CIC with the desired properties.

In our arguments below, we will always show that $\tilde{C}$ satisfies the two properties that were just mentioned.

As a first step, we now establish that C and $\hat{C}$ "intersect".

Claim 4. $O(C) \cap O(\hat{C}) \neq \emptyset$

Now we complete the proof of Lemma 3 for the case of $I(C) \cap I(\hat{C}) = \{j^*\}$. Let l be such that $j^l = j^*$. Note that $I(C) \cap I(\hat{C}) = \{j^*\}$ and Claim 4 imply $O(C) \cap O(\hat{C}) = \{o^1\}$ as well as $p^{l-1} = o^1$. If $j^* = \hat{j}^*$, we get a contradiction since $p^L \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$ (as $\hat{C}$ is a CIC for $\hat{j}^*$) and $o^1 \in A_{j^*}$ (as C is a CIC for j^*) imply $p^L \neq o^1$. If $j^* \neq \hat{j}^*$, then we must have $j^* < \hat{j}^*$. Since $\hat{C}$ is a CIC for $\hat{j}^*$, we obtain that $p^l \in A_{j^*}$. Given that $I(C) \cap I(\hat{C}) = \{j^*\}$ and $O(C) \cap O(\hat{C}) = \{o^1\}$, it is then easy to verify that

$$(j^l, p^l, \dots [p^L, j^1] \dots, j^{l-1}, p^{l-1}, i^2, o^2, \dots, i^M, o^M)$$

is a CIC of μ for j^* at A that makes all agents exactly as well off as $(\mu + C) + \hat{C}$. This completes of the desired statement in case $I(C) \cap I(\hat{C}) = \{j^*\}$.

Henceforth, we assume that $I(C) \cap I(\hat{C}) \neq \{j^*\}$. Note that Claim 4 then implies $(I(C) \cap I(\hat{C})) \setminus \{j^*\} \neq \emptyset$.

We now introduce some additional notation for the remainder of the proof. First, given two integers $m, m' \in \{1, \dots, M\}$, we use the notation

- $\{m, \dots [M, 1] \dots, m'\}$ to represent $\{m, \dots, m'\}$ if $m \leq m'$ and $\{1, \dots, m'\} \cup \{m, \dots, M\}$ if $m > m'$,
- $(i^m, o^m, \dots [o^M, i^1] \dots, i^{m'}, o^{m'})$ to represent $(i^m, o^m, \dots, i^{m'}, o^{m'})$ if $m \leq m'$ and $(i^m, o^m, \dots, i^M, o^M, i^1, o^1, \dots, i^{m'}, o^{m'})$ if $m > m'$,
- $(i^m, \dots [i^M, i^1] \dots, i^{m'})$ to represent $(i^m, \dots, i^{m'})$ if $m \leq m'$ and $(i^m \dots, i^M, i^1, \dots, i^{m'})$ if $m > m'$, and
- $(o^m, \dots [o^M, o^1] \dots, o^{m'})$ to represent $(o^m, \dots, o^{m'})$ if $m \leq m'$ and $(o^m \dots, o^M, o^1, \dots, o^{m'})$ if $m > m'$.

Analogously, given two integers $l, l' \in \{1, \dots, L\}$, we use the notations $\{l, \dots [L, 1] \dots, l'\}$, $(j^n, p^l, \dots [p^L, j^1] \dots, j^{l'}, p^{l'})$, $(j^l, \dots [j^L, j^1] \dots, j^{l'})$, and $(p^l, \dots [p^L, p^1] \dots, p^{l'})$.

Second, let $\underline{l}^*$ be the smallest integer such that either $j^{\underline{l}^*} \in I(C) \setminus \{\hat{j}^*\}$ or $p^{\underline{l}^*} \in O(C)$. Claim 4 ensures that such $\underline{l}^*$ exists. We now use $\underline{l}^*$ to define a specific point on the cycle C. In doing so, we distinguish two cases:

1. If $j^{\underline{l}^*} \notin I(C) \setminus \{\hat{j}^*\}$ and $p^{\underline{l}^*} \in O(C)$, then let $\underline{l} = \underline{l}^*$ and let $m(\underline{l})$ be such that $p^{\underline{l}} = o^{m(\underline{l})-1}$ (where $o^0 \equiv o^M$).

2. If $j^{\underline{l}^*} \in I(C) \setminus \{\hat{j}^*\}$, then let $\underline{l} = \underline{l}^* - 1$ and let $m(\underline{l})$ be such that $j^{\underline{l}^*} = i^{m(\underline{l})}$.[13]

Note that in both of the possible two cases, our definition of $\underline{l}$ ensures that $\underline{l} \geq 1$, $j^{\underline{l}} \notin I(C) \setminus \{\hat{j}^*\}$, and $p^{\underline{l}} \in \mu(i^{m(\underline{l})})$. To understand the significance of $m(\underline{l})$, note first that the definition of $\underline{l}$ implies that $\{j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}\} \cap \{i^1, o^1, \ldots, i^M, o^M\} \subseteq \{j^1, p^{\underline{l}}\}$. Now consider the following sequence of agent-object pairs:

$$\underline{C} \equiv (j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^M, o^M).$$

In $\underline{C}$, each agent points to an object that she does not get at μ and each object, except possibly o^M, points to the agent who gets it at μ. Furthermore, except possibly j^1, all agents and objects in $\underline{C}$ are distinct from each other. These facts about $\underline{C}$ are useful for our construction of CICs in the remainder of the proof.

Third, let $\bar{l}$ be the least integer such that either $j^{\bar{l}} \in I(C) \setminus \{\hat{j}^*\}$ or $p^{\bar{l}} \in O(C)$. Again, Claim 4 ensures that such $\bar{l}$ exists. As is the case for $\underline{l}^*$, we use $\bar{l}$ to define a specific point of the cycle C. Before doing that, we discuss the case of $p^{\bar{l}} \in O(C)$ in more detail by means of the following claim.

Claim 5. *If $p^{\bar{l}} \in O(C)$, then $\bar{l} = L$.*

Now consider the case where $p^{\bar{l}} \notin O(C)$ and $j^{\bar{l}} \in I(C) \setminus \{\hat{j}^*\}$. Let $m(\bar{l})$ be such that $i^{m(\bar{l})+1} = j^{\bar{l}}$.[14] Since C is a CIC of μ, $j^{\bar{l}} = i^{m(\bar{l})+1}$ implies that $o^{m(\bar{l})} \in \mu(j^{\bar{l}})$. By definition of $\bar{l}$, we have that $\{i^1, o^1, \ldots, i^M, o^M\} \cap \{j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L\} = \{j^{\bar{l}}\}$. Now let $m' \in \{1, \ldots, M\}$ be an arbitrary integer $\neq m(\bar{l}) + 1$ and consider the following sequence of agent-object pairs:

$$\overline{C} \equiv (i^{m'}, o^{m'}, \ldots [o^M, i^1] \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L).$$

In $\overline{C}$, each agent points to an object that she does not get at μ and each object, except possibly p^L, points to the agent who gets it at μ. Furthermore, all agents and objects in $\overline{C}$ are distinct from each other. Again, these facts are useful for our construction of CICs in the remainder of the proof.

Fourth, let $\underline{m}$ be the smallest integer m such that $i^m \in I(\hat{C}) \setminus \{j^*\}$, i.e. $i^{\underline{m}} \in I(\hat{C}) \setminus \{j^*\}$ and for each $m \in \{1, \ldots, \underline{m} - 1\}$, $i^m \notin I(\hat{C}) \setminus \{j^*\}$. Note that such an integer exists because $(I(C) \cap I(\hat{C})) \setminus \{j^*\} \neq \emptyset$. Similarly, let $\overline{m}$ be the largest integer m' such that $i^{m'} \in I(\hat{C}) \setminus \{j^*\}$,

[13] Note that $\underline{l}^* \geq 2$ in this case.

[14] See Figure 1 for the case where $j^{\underline{l}} \notin I(C) \setminus \{\hat{j}^*\}$ and $p^{\bar{l}} \notin O(C)$.

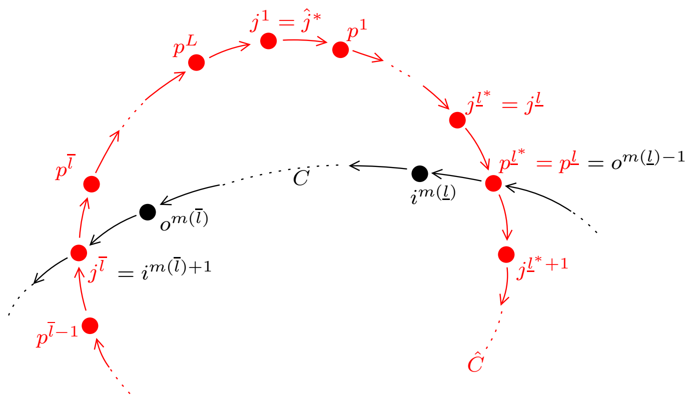


Figure 1: Definitions of $\underline{l}^*, \underline{l}, \bar{l}, m(\underline{l})$, and $m(\bar{l})$ for the case where $j^{\underline{l}^*} \notin I(C) \setminus \{j'\}$ and $p^{\bar{l}} \notin O(C)$.

i.e. $i^{\overline{m}} \in I(\hat{C}) \setminus \{j^*\}$ and for each $m \in \{\overline{m}+1, \ldots, M\}$, $i^m \notin I(\hat{C}) \setminus \{j^*\}$. Note that it is possible that $\overline{m} = \underline{m}$ and that in this case, we have $(I(C) \cap I(\hat{C})) \setminus \{j^*\} = \{i^{\overline{m}}\}$, $o^{\overline{m}} \in O(\hat{C})$, and $O(C) \cap O(\hat{C}) \subseteq \{o^1, o^{\overline{m}}\}$. Finally, let m^* be such that $i^{m^*} = i$ and $o^{m^*} = o$. Note that the existence of such an m^* follows since $o \in (\mu + C)(i) \setminus \mu(i)$ by Claim 2.

Our next claim is that objects and agents that show up in both C and $\hat{C}$ must either all appear "before", or all appear "after" agent i in C, starting from $i^1 (= j^*)$.

Claim 6. *$o \notin O(\hat{C})$ and if $\underline{m} \leq m^*$, then $\overline{m} \leq m^*$.*

The next three claims lay the foundation for splicing together parts of C and $\hat{C}$ to form new CICs. The main hurdle is that we need to ensure that agents with higher priority than $\hat{j}^*$ not only are unaffected by both C and $\hat{C}$, but that they trade only like objects: either desirable for desirable or undesirable endowment for undesirable endowment. This allows us to cut and paste parts of C and $\hat{C}$ together to construct the CIC equivalent to the sum of C and $\hat{C}$ from the perspective of any agent with priority at least as high as $\hat{j}^*$. The first of these claims pertains to agents in $I(C)$ who have higher priority than $\hat{j}^*$ and appear in C "sandwiched" between two agents in $I(\hat{C})$.

Claim 7. *Assume that $\underline{m} \neq \overline{m}$.*

1. *For each $m' \in \{\underline{m}+1, \ldots, \overline{m}-1\}$, if $i^{m'} < \hat{j}^*$, either $\{o^{m'-1}, o^{m'}\} \subseteq A_{i^{m'}}$ or $\{o^{m'-1}, o^{m'}\} \subseteq \Omega_{i^{m'}} \setminus A_{i^{m'}}$. If, additionally, there is l' such that $j^{l'} = i^{m'}$, then either $\{o^{m'-1}, o^{m'}, p^{l'-1}, p^{l'}\} \subseteq A_{i^{m'}}$ or $\{o^{m'-1}, o^{m'}, p^{l'-1}, p^{l'}\} \subseteq \Omega_{i^{m'}} \setminus A_{i^{m'}}$.*

2. *If $i^{\underline{m}} < \hat{j}^*$ and $l(\underline{m})$ is such that $j^{l(\underline{m})} = i^{\underline{m}}$, then either $\{o^{\underline{m}}, p^{l(\underline{m})-1}, p^{l(\underline{m})}\} \subseteq A_{i^{\underline{m}}}$ or $\{o^{\underline{m}}, p^{l(\underline{m})-1}, p^{l(\underline{m})}\} \subseteq \Omega_{i^{\underline{m}}} \setminus A_{i^{\underline{m}}}$.*

3. *If $i^{\overline{m}} < \hat{j}^*$ and $l(\overline{m})$ is such that $j^{l(\overline{m})} = i^{\overline{m}}$, then either $\{o^{\overline{m}-1}, p^{l(\overline{m})-1}, p^{l(\overline{m})}\} \subseteq A_{i^{\overline{m}}}$ or $\{o^{\overline{m}-1}, p^{l(\overline{m})-1}, p^{l(\overline{m})}\} \subseteq \Omega_{i^{\overline{m}}} \setminus A_{i^{\overline{m}}}$.*

Our second claim considers agents in $I(C) \setminus I(\hat{C})$ who have priority than $\hat{j}^*$ but, when they appear in C, are not sandwiched between two agents in $I(\hat{C})$. The next claim pertains to the case where j^* trades one desirable object for another in $\hat{C}$.

Claim 8. *Assume that there exists an l^* such that $j^* = j^{l^*}$ and that $\{p^{l^*-1}, p^{l^*}\} \subseteq A_{j^*}$. There does not exist an integer $\check{m} \in \{2, \ldots, \underline{m}\}$ such that $i^{\check{m}} < \hat{j}^*$ and either*

1. *$o^{\check{m}-1} \in A_{i^{\check{m}}}$ as well as $o^{\check{m}} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$, or*
2. *$o^{\check{m}-1} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$ as well as $o^{\check{m}} \in A_{i^{\check{m}}}$.*

Our third claim considers the mirror image of what is considered in Claim 8: the case where j^* trades two undesirable endowed objects in C.

Claim 9. *Assume that there exists an l^* such that $j^* = j^{l^*}$ and that $\{p^{l^*-1}, p^{l^*}\} \subseteq \Omega_{j^*} \setminus A_{j^*}$. There does not exist an integer $\check{m} \in \{\overline{m}, \ldots, M\}$ such that $i^{\check{m}} < \hat{j}^*$ and either*

1. *$o^{\check{m}-1} \in A_{i^{\check{m}}}$ as well as $o^{\check{m}} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$, or*
2. *$o^{\check{m}-1} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$ as well as $o^{\check{m}} \in A_{i^{\check{m}}}$.*

We now use the above claims to show that there is a CIC $\tilde{C}$ of μ for j^* at A that satisfies (P1) - (P3). Clearly, it would be impossible to find a CIC that satisfies (P2) if C and $\hat{C}$ both increase the welfare of $\hat{j}^*$: Since all agents in a CIC are distinct, a CIC can increase the welfare of any agent by at most one. Our final auxiliary claim shows that the situation mentioned in the previous situation is impossible.

Claim 10. *C cannot increase the welfare of $\hat{j}^*$*

Note that since C is a CIC for j^*, Claim 10 immediately implies $j^* \neq \hat{j}^*$. In the following, we distinguish four cases according to the role, if any, that $\hat{j}^*$ plays in C.

Case 1: C decreases the welfare of $\hat{j}^*$. If C decreases the welfare of $\hat{j}^*$, then there exists an m' such that $i^{m'} = \hat{j}^*$, $o^{m'-1} \in A_{\hat{j}^*}$, and $o^{m'} \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$. Since $j^* \neq \hat{j}^*$ we must have $m' \geq 2$. We distinguish two subcases.

Case 1.1 $\{o^1, i^2, o^2, \ldots, i^{m'-1}, o^{m'-1}\} \cap \{j^1, p^1, \ldots, j^L, p^L\} = \emptyset$ We split our arguments into three further subcases:

Case 1.1.1: j^* trades desirable objects in $\hat{C}$. In this subcase, there exists an $\tilde{l} > 1$ such that $j^{\tilde{l}} = j^*$ and $\{p^{\tilde{l}-1}, p^{\tilde{l}}\} \subseteq A_{j^*}$. Let $\tilde{l}'$ be the first integer in $(\tilde{l}, \ldots, L)$ such that either $j^{\tilde{l}'} \in I(C) \setminus \{j^*, \hat{j}^*\}$ or $p^{\tilde{l}'} \in O(C) \cup \{p^L\}$.
If $j^{\tilde{l}'} \in I(C) \setminus \{j^*, \hat{j}^*\}$, let $\tilde{m}'$ be such that $i^{\tilde{m}'} = j^{\tilde{l}'}$ and consider

$$\tilde{C} \equiv (j^{\tilde{l}}, p^{\tilde{l}}, \ldots, j^{\tilde{l}'-1}, p^{\tilde{l}'-1}, i^{\tilde{m}'}, o^{\tilde{m}'}, \ldots, i^M, o^M).$$

Note that we must have $\tilde{l}' - 1 \geq \tilde{l}$ since $j^{\tilde{l}'} \neq j^* = j^{\tilde{l}}$. Given that $p^{\tilde{l}} \in A_{j^*}$ and $o^M \in \Omega_{j^*} \setminus A_{j^*}$, it is straightforward to show that $\tilde{C}$ is a CIC of μ for j^* at A. The assumptions that $\{i^2, \ldots, i^{m'-1}\} \cap I(C) = \emptyset$ and $i^{\tilde{m}'} = j^{\tilde{l}'} \neq j^* = i^1$ jointly imply that $\tilde{m}' \geq m'$. Since $\tilde{l}' > \tilde{l} > 1$, we have that $j^{\tilde{l}'} \neq j^1 = \hat{j}^*$. Combining the previous two observations, we obtain that $\tilde{m}' > m'$ and $\hat{j}^* \notin I(\tilde{C})$. By Claims 7 and 8, there cannot be an $m \in \{2, \ldots, \tilde{m}' - 1\}$ such that $i^m < \hat{j}^*$ and such that C increases the welfare of i^m. Since $\overline{m} \geq \tilde{m}' > m' \geq \underline{m}$, Claim 7 implies that if $i^{\tilde{m}'} < \hat{j}^*$, then $o^{\tilde{m}'-1} \in \Omega_{i^{\tilde{m}'}} \setminus A_{i^{\tilde{m}'}}$ is possible only if $p^{\tilde{l}'-1} \in \Omega_{i^{\tilde{m}'}} \setminus A_{i^{\tilde{m}'}}$ as well. Hence, if $i^{\tilde{m}'} < \hat{j}^*$, then $\tilde{C}$ increases the welfare of $i^{\tilde{m}'}$ whenever C increases the welfare of $i^{\tilde{m}'}$. Combing the previous arguments, we obtain that all agents $j' \leq \hat{j}^*$ are at least well off under $\mu + \tilde{C}$ as under $(\mu + C) + \hat{C}$. Finally, note that the definitions of $\tilde{l}$ and $\tilde{l}'$ imply that either $i \notin I(\tilde{C})$ or i receives $o \in A_i$ in $\tilde{C}$. Hence, $\tilde{C}$ weakly increases the welfare of i.
If $j^{\tilde{l}'} \notin I(C) \setminus \{j^*\}$, let $\tilde{m}'$ be such that $p^{\tilde{l}'} = o^{\tilde{m}'-1}$ and consider

$$\tilde{C}' \equiv (j^{\tilde{l}}, p^{\tilde{l}}, \ldots, j^{\tilde{l}'}, p^{\tilde{l}'}, i^{\tilde{m}'}, o^{\tilde{m}'}, \ldots, i^M, o^M).$$

Given that $\{o^1, \ldots, o^{m'-1}\} \cap O(C) = \emptyset$, we must have $\tilde{m}' - 1 \geq m'$. Since $p^L \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$, $\tilde{C}'$ can therefore not decrease the welfare of $\hat{j}^*$. An analog of the arguments where $j^{\tilde{l}'} \in I(C) \setminus \{j^*\}$ shows that $\tilde{C}'$ is a CIC of μ for j^* at A that has the desired properties.

Case 1.1.2: j^* trades undesirable endowed objects in $\hat{C}$. In this subcase, there exists an $\tilde{l} > 1$ such that $j^{\tilde{l}} = j^*$ and $\{p^{\tilde{l}-1}, p^{\tilde{l}}\} \subseteq \Omega_{j^*} \setminus A_{j^*}$. Note that we must have $p^{\tilde{l}-1} \notin O(C)$ since C is a CIC in which $j^* = i^1$ receives the desirable object $o^1 \in A_{j^*}$. Let $\tilde{l}'$ be the first integer in $(1, \ldots, \tilde{l} - 1)$ such that either $j^{\tilde{l}'} \in I(C) \setminus \{j^*, \hat{j}^*\}$ or $p^{\tilde{l}'} \in O(C) \cup \{p^{\tilde{l}-1}\}$.
If $j^{\tilde{l}'} \in I(C) \setminus \{j^*, \hat{j}^*\}$, let $\tilde{m}'$ be such that $i^{\tilde{m}'} = j^{\tilde{l}'}$ and consider

$$\tilde{C} \equiv (i^1, o^1, \ldots, i^{m'-1}, o^{m'-1}, j^1, p^1, \ldots, j^{\tilde{l}'-1}, p^{\tilde{l}'-1}, i^{\tilde{m}'}, o^{\tilde{m}'}, \ldots, i^M, o^M).$$

Since $\{o^1, i^2, o^2, \ldots, i^{m'-1}, o^{m'-1}\} \cap \{j^1, p^1, \ldots, j^L, p^L\} = \emptyset$, we have that $\tilde{m}' > m'$ and $o^{m'-1} \neq p^1$. It is then straightforward to show that $\tilde{C}$ is a CIC of μ for j^* at A. Since $\underline{m} \leq m' < \tilde{m}' \leq \overline{m}$, Claim 7 implies that there does not exist an $m \in \{m', \ldots, \tilde{m}' - 1\}$ such that $i^m < \hat{j}^*$ and such that C increases the welfare of i^m. Since $\tilde{m}' > \underline{m}$, Claim 7 implies that if $i^{\tilde{m}'} < \hat{j}^*$, then $o^{\tilde{m}'-1} \in \Omega_{i^{\tilde{m}'}} \setminus A_{i^{\tilde{m}'}}$ is possible only if $p^{\tilde{l}'-1} \in \Omega_{i^{\tilde{m}'}} \setminus A_{i^{\tilde{m}'}}$ as well. Hence, if $i^{\tilde{m}'} < \hat{j}^*$, then $\tilde{C}$ increases the welfare of $i^{\tilde{m}'}$ whenever C increases the welfare of $i^{\tilde{m}'}$. Combining the previous arguments, we obtain that all agents $j' \leq \hat{j}^*$ are at least well off under $\mu + \tilde{C}$ as under $(\mu + C) + \hat{C}$. The definitions of $\tilde{l}'$ and $\tilde{m}'$ imply that either $i \notin I(\tilde{C})$ or that i receives the desirable object o in $\tilde{C}$. Hence, $\tilde{C}$ weakly increases the welfare of i.
If $\tilde{l}' = \tilde{l} - 1$ and $j^{\tilde{l}-1} \notin I(C) \setminus \{j^*, \hat{j}^*\}$, then consider

$$\tilde{C}' \equiv (i^1, o^1, \ldots, i^{m'-1}, o^{m'-1}, j^1, p^1, \ldots, j^{\tilde{l}-1}, p^{\tilde{l}-1}).$$

Since $p^{\tilde{l}-1} \in \Omega_{j^*} \setminus (A_{j^*} \cup O(C))$, analogs of our earlier arguments show that $\tilde{C}'$ is a CIC of μ for j^* at A. Since j^* trades two undesirable objects in $\hat{C}$ and since $m' = \underline{m}$, Claims 7 and 9 imply that there cannot be an agent $j' \in \{i^{m'+1}, \ldots, i^M\}$ such that $j' < \hat{j}^*$ and such that C improves the welfare of j'. Combining our previous arguments, we find that all agents $j \leq \hat{j}^*$ are at least as well off under $\mu + \tilde{C}'$ as under $(\mu + C) + \hat{C}$. As before, i cannot do worse than receiving the desirable object o in $\tilde{C}'$ so that $\tilde{C}'$ weakly increases the welfare of i.
Next, assume that $\tilde{l}' < \tilde{l} - 1$ and $j^{\tilde{l}'} \notin I(C) \setminus \{j^*, \hat{j}^*\}$. Since $\tilde{l}' < \tilde{l}$, we must have $j^{\tilde{l}'} \neq j^*$. Since $\tilde{l}' < \tilde{l} - 1$, the definition of $\tilde{l}'$ implies that we must have $p^{\tilde{l}'} \in O(C)$. Let $\tilde{m}'$ be such that $p^{\tilde{l}'} = o^{\tilde{m}'-1}$ and note that $\{o^1, \ldots, o^{m'-1}\} \cap O(\hat{C}) = \emptyset$ implies $\tilde{m}' - 1 \geq m'$. Since $\tilde{l}' < \tilde{l} - 1 < L$, we must have $j^{\tilde{l}'+1} \neq \hat{j}^* = j^1$ and hence obtain that $\tilde{m}' - 1 > m'$ as well as $j^{\tilde{l}'+1} = i^{\tilde{m}'-1} \in I(C) \setminus \{j^*, \hat{j}^*\}$. Now consider

$$\tilde{C}'' \equiv (i^1, o^1, \ldots, i^{m'-1}, o^{m'-1}, j^1, p^1, \ldots, j^{\tilde{l}'}, p^{\tilde{l}'}, i^{\tilde{m}'}, o^{\tilde{m}'}, \ldots, i^M, o^M).$$

Again, analogs of our earlier arguments show that $\tilde{C}''$ is a CIC of μ for j^* at A. Since $\underline{m} = m' < \tilde{m}' - 1 \leq \overline{m}$, Claim 7 and the conditions of Case 1.1.2. again imply that all agents $j' \leq \hat{j}^*$ are at least as well off under $\mu + \tilde{C}''$ as under $(\mu + C) + \hat{C}$. Finally, i either receives o in $\tilde{C}''$ or $i \notin I(\tilde{C}'')$ so that $\tilde{C}''$ weakly increases the welfare of i.

Case 1.1.3: j^* does not participate in $\hat{C}$. We argue first that $i^{m(\underline{l})} \neq \hat{j}^*$: otherwise, $p^{\underline{l}} \in \mu(i^{m(\underline{l})})$ and $o^{m'-1} \notin O(\hat{C})$ (recall that the latter holds since we assume $\{o^1, \ldots, o^{m'-1}\} \cap O(\hat{C}) = \emptyset$ in Case 1.1.) would imply that $p^{\underline{l}} \notin O(C)$; since $p^{\underline{l}} \in \mu(\hat{j}^*) \setminus O(C)$ is possible only if $\underline{l} = L$ and $O(C) \cap O(\hat{C}) = \emptyset$, we obtain a contradiction to Claim 4.

Next, assume that $p^{\underline{l}} = o^M$ and consider

$$\tilde{C} \equiv (i^1, o^1, \ldots, i^{m'-1}, o^{m'-1}, j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}).$$

By analogs of our earlier arguments, $\tilde{C}$ is a CIC of μ for j^* at A that weakly increases the welfare of i. Furthermore, we must have $m^* \leq m' - 1$ since otherwise $\tilde{C}$ is a CIC of μ for j^* at $\hat{A}$, thus contradicting $\mu \in \mathcal{M}^{i-1}(\hat{A})$. By Claim 7, there does not exist an $m \in \{m', \ldots, \overline{m} - 1\}$ such that $i^m < \hat{j}^*$ and such that C increases the welfare of i^m. We complete the proof of Case 1.1.3 in case $p^{\underline{l}} = o^M$ by showing that there also does not exist an index $\check{m} \geq \overline{m}$ such that $i^{\check{m}} < \hat{j}^*$ and such that C increases the welfare of $i^{\check{m}}$. Assume to the contrary that such an $\check{m}$ exists and also assume, without loss of generality, that there is no $m \in \{\overline{m}, \ldots, M\}$ such that $i^m < i^{\check{m}}$ and such that C improves the welfare of i^m. Let $\hat{l}$ be the first integer in $(\underline{l} + 1, \ldots [L, 1] \ldots, \underline{l})$ such that $j^{\hat{l}} \in I(C)$. If $j^{\hat{l}} = i^{\check{m}}$, we must have $\check{m} = \overline{m}$ and the third part of Claim 7 implies $p^{\hat{l}-1} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$. Furthermore, we must have $p^{\hat{l}-1} \notin O(C)$ given that $o^{\check{m}} \in A_{i^{\check{m}}}$ implies $o^{\check{m}} \neq p^{\hat{l}-1}$ and $\hat{l} \geq \underline{l} + 1$. Since $m^* < m' < \check{m}$ and $p^{\underline{l}} = o^M$, our definition of $\check{m}$ implies that

$$(i^{\check{m}}, o^{\check{m}}, \ldots, i^{M-1}, o^{M-1}, j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^{\hat{l}-1}, p^{\hat{l}-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$, thus contradicting $\mu \in \mathcal{M}^{i-1}(\hat{A})$. If $j^{\hat{l}} \neq i^{\check{m}}$ and $p^{\hat{l}-1} \notin O(C)$, let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. Since

$$(i^{\hat{m}}, o^{\hat{m}}, \ldots, i^{M-1}, o^{M-1}, j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^{\hat{l}-1}, p^{\hat{l}-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$, we obtain a contradiction as before. Finally, if $j^{\hat{l}} \neq i^{\check{m}}$ and $p^{\hat{l}-1} \in O(C)$, let $\hat{m}$ be such that $o^{\hat{m}-1} = p^{\hat{l}-1}$. Given that $\hat{m} - 1 \geq \underline{m}$ (as $j^{\hat{l}} = i^{m(\underline{l})-1}$ and $j^* \notin I(\hat{C})$ in this case), we again find that

$$(i^{\hat{m}}, o^{\hat{m}}, \ldots, i^{M-1}, o^{M-1}, j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^{\hat{l}-1}, p^{\hat{l}-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$.

For the remainder of Case 1.1.3, we assume that $p^{\underline{l}} \neq o^M$. Since $j^* \notin I(\hat{C})$, we have $i^{m(\underline{l})} \neq j^*$ and $m(\underline{l}) \geq \underline{m}$. Furthermore,

$$\tilde{C}' \equiv (i^1, o^1, \ldots, i^{m'-1}, o^{m'-1}, j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at A that weakly increases the welfare of i. Since $\mu \in \mathcal{M}^{i-1}(\hat{A})$, we must have $m^* \in \{1, \ldots, m'-1\} \cup \{m(\underline{l}), \ldots, M\}$. If $m(\underline{l}) \leq \overline{m}$, Claim 7 implies that $\tilde{C}'$ is a CIC of μ for j^* at A such that all agents $j \leq \hat{j}^*$ are at least as well off under $\mu + \tilde{C}'$ as under $(\mu + C) + \hat{C}$. If $m(\underline{l}) > \overline{m}$, then we must have $m(\underline{l}) - 1 = \overline{m}$ and $j^{\underline{l}+1} = i^{\overline{m}}$. We claim that if $i^{\overline{m}} < \hat{j}^*$, then C cannot increase the welfare of $i^{\overline{m}}$. Suppose the contrary. The third part of Claim 7 implies that $p^{\underline{l}+1} \in A_{i^{\overline{m}}}$. As in the case of $p^{\underline{l}} = o^M$, let $\hat{l}$ be the first integer in $(\underline{l}+1, \ldots [L, 1] \ldots, \underline{l})$ such that $j^{\hat{l}} \in I(C)$. If $p^{\hat{l}-1} \notin O(C)$, let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. Since $m'-1 \leq \hat{m} \leq m(\underline{l}) - 1 < m(\underline{l})$, we have that $m^* \notin \{\hat{m}, \ldots, m(\underline{l}) - 1\}$. Hence,

$$(i^{\hat{m}}, o^{\hat{m}}, \ldots, i^{m(\underline{l})-1}, o^{m(\underline{l})-1}, j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^{\hat{l}-1}, p^{\hat{l}-1})$$

is a CIC of μ for $i^{\overline{m}}$ at $\hat{A}$, which is impossible. If $p^{\hat{l}-1} \in O(C)$, let $\hat{m}$ be such that $o^{\hat{m}-1} = p^{\hat{l}-1}$. Since $\{o^1, \ldots, o^{m'-1}\} \cap O(C) = \emptyset$, we obtain that $\hat{m} - 1 \geq m'$. Again, we obtain a contradiction since

$$(i^{\hat{m}}, o^{\hat{m}}, \ldots, i^{m(\underline{l})-1}, o^{m(\underline{l})-1}, j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^{\hat{l}-1}, p^{\hat{l}-1})$$

is a CIC of μ for $i^{\overline{m}}$ at $\hat{A}$, which is impossible.

Case 1.2: $\boldsymbol{\{o^1, i^2, o^2, \ldots, i^{m'-1}, o^{m'-1}\} \cap \{j^1, p^1, \ldots, j^L, p^L\} \neq \emptyset}$. Let $\tilde{l}$ be the last integer in $(1, \ldots, L)$ such that either $j^{\tilde{l}} \in \{i^2, \ldots, i^{m'-1}\}$ or $p^{\tilde{l}} \in \{o^1, \ldots, o^{m'-1}\}$. We claim that $p^{\tilde{l}} \notin \{o^1, \ldots, o^{m'-1}\}$: if $\tilde{l} = L$ and $p^{\tilde{l}} \in \{o^1, \ldots, o^{m'-1}\}$, we obtain a contradiction to $p^L \in (\mu + C)(\hat{j}^*)$ since $o^m \notin (\mu + C)(\hat{j}^*)$ for all $m \leq m' - 1$ given that $i^{m'} = \hat{j}^*$ and all agents in $I(C)$ are distinct; if $\tilde{l} < L$ and $p^{\tilde{l}} \in \{o^1, \ldots, o^{m'-1}\}$, we obtain $j^{\tilde{l}+1} \in \{i^1, \ldots, i^{m'-1}\}$, thus contradicting the definition of $\tilde{l}$. Hence, we must have $p^{\tilde{l}} \notin \{o^1, \ldots, o^{m'-1}\}$ and thus $j^{\tilde{l}} \in \{i^2, \ldots, i^{m'-1}\}$. In the following, let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$.

Note that we must have $i \neq i^{\tilde{m}} = j^{\tilde{l}}$ since $\underline{m} \leq \tilde{m} < m' \leq \overline{m}$ and Claim 6 imply $m^* \neq \tilde{m}$. The observation that $i \neq j^{\tilde{l}}$ will allow us to easily infer that each time we construct a CIC of μ for j^* at A that makes all agents $j' \leq \hat{j}^*$ at least as well off as $(\mu + C) + \hat{C}$, the CIC also weakly increases the welfare of i.

Next, let $\tilde{l}'$ be the first integer in $(\tilde{l}, \dots, L, 1)$ such that either $j^{\tilde{l}'} \in \{i^{m'}, \dots, i^M\} \cup \{j^*\}$ or $p^{\tilde{l}'} \in \{o^{m'}, \dots, o^M\} \cup \{p^L\}$.

Note that we allow for the possibility that $\tilde{l}' = \tilde{l}$, in which case we would have $p^{\tilde{l}} \in \{o^{m'}, \dots, o^M\} \cup \{p^L\}$ (since $j^{\tilde{l}} \in \{i^2, \dots, i^{m'-1}\}$) and $\{j^{\tilde{l}+1}, p^{\tilde{l}+1}, \dots, j^{\tilde{l}'-1}, p^{\tilde{l}'-1}\} = \emptyset$. Note also that the existence of a $\tilde{l}'$ with the desired property is ensured since we allow for the possibility that $\tilde{l}' = L$, $j^L \notin I(C)$, and $p^L \notin O(C)$. We distinguish four further subcases:

Case 1.2.1: $\boldsymbol{j^{\tilde{l}'} \neq j^*}$ and $\boldsymbol{j^{\tilde{l}'} \in \{i^{m'}, \dots, i^M\}}$. Let $\tilde{m}'$ be such that $i^{\tilde{m}'} = j^{\tilde{l}'}$ and consider

$$\tilde{C} \equiv (i^1, o^1, \dots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \dots, j^{\tilde{l}'-1}, p^{\tilde{l}'-1}, i^{\tilde{m}'}, o^{\tilde{m}'}, \dots, i^M, o^M).$$

Note that we must have $\tilde{l}' > \tilde{l}$ since $j^{\tilde{l}} \in \{i^2, \dots, i^{m'-1}\}$ and $j^{\tilde{l}'} \in \{i^{m'}, \dots, i^M\}$. Since $\tilde{l} \geq 1$, $\tilde{l}' > 1$ and therefore $j^{\tilde{l}'} \neq \hat{j}^*$, which in turn implies $\tilde{m}' \neq m'$. From this last observation, since $\tilde{m} \in \{2, \dots, m'-1\}$ and $\tilde{m}' \in \{m', \dots, M\}$, we have $\tilde{m} - 1 < m' < \tilde{m}'$. Hence, we obtain $\hat{j}^* \notin I(\tilde{C})$. Given that $\underline{m} \leq \tilde{m} < \tilde{m}' \leq \overline{m}$, Claim 7 implies that $\tilde{C}$ is a CIC of μ for j^* at A that makes all agents $j' \leq \hat{j}^*$ weakly better off than $(\mu + C) + \hat{C}$. Furthermore, $j^{\tilde{l}} \neq i$ and the definition of $\tilde{l}'$ are easily seen to imply that $\tilde{C}$ weakly increases the welfare of i.

Case 1.2.2: $\boldsymbol{j^{\tilde{l}'} \notin \{i^{m'}, \dots, i^M\} \cup \{j^*\}}$ and $\boldsymbol{p^{\tilde{l}'} \in \{o^{m'}, \dots, o^M\}}$. Let $\tilde{m}'$ be such that $o^{\tilde{m}'-1} = p^{\tilde{l}'}$. Then $\tilde{m}' - 1 \geq m'$. Furthermore, since $j^{\tilde{l}} \in \{i^2, \dots, i^{m'-1}\}$, $\tilde{m} \leq m' - 1$. Therefore, $\hat{j}^* = i^{m'} \notin \{i^1, \dots, i^{\tilde{m}-1}, i^{\tilde{m}'}, \dots, i^M\}$. Now consider

$$\tilde{C} \equiv (i^1, o^1, \dots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \dots, j^{\tilde{l}'}, p^{\tilde{l}'}, i^{\tilde{m}'}, o^{\tilde{m}'}, \dots, i^M, o^M).$$

If $\tilde{m}' \leq \overline{m}$, Claim 7 is easily seen to imply that $\tilde{C}$ is a CIC of μ for j^* at A that makes all agents $j' \leq \hat{j}^*$ weakly better off then $(\mu + C) + \hat{C}$ and that weakly increases the welfare of i.

If $\tilde{m}' > \overline{m}$, we must have $\tilde{m}' - 1 = \overline{m}$ given that $j^{\tilde{l}'+1} = i^{\tilde{m}'-1}$ and $\tilde{m}' - 1 \geq m' > 1$.

Assume first that $i^{\overline{m}} < \hat{j}^*$, $o^{\overline{m}-1} \in \Omega_{i^{\overline{m}}} \setminus A_{i^{\overline{m}}}$, and $o^{\overline{m}} \in A_{i^{\overline{m}}}$. The third part of Claim 7 implies $p^{\tilde{l}'} \in \Omega_{i^{\overline{m}}} \setminus A_{i^{\overline{m}}}$. However, $p^{\tilde{l}'} \in (\mu + C)(i^{\overline{m}})$ and $p^{\tilde{l}'} \in \{o^{m'}, \dots, o^M\}$ imply that $p^{\tilde{l}'} = o^{\overline{m}}$, which is impossible given that $o^{\overline{m}} \in A_{i^{\overline{m}}}$.

Hence, C cannot improve the welfare of $i^{\overline{m}}$ if $i^{\overline{m}} < \hat{j}^*$. As in the case of $\tilde{m}' \le \overline{m}$, one can then verify that $\tilde{C}$ is a CIC with the desired properties.

Case 1.2.3: $\tilde{l}' = L$, $j^L \notin \{i^{m'}, \ldots, i^M\} \cup \{j^*\}$, and $p^L \notin \{o^{m'}, \ldots, o^M\}$. Note that since $p^L \in (\mu + C)(\hat{j}^*)$ and $i^{m'} = \hat{j}^*$, $p^L \notin \{o^{m'}, \ldots, o^M\}$ implies $p^L \notin O(C)$. Given that $\{p^L, o^{m'}\} \subseteq \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$ and $\underline{m} \le \tilde{m} < m' \le \overline{m}$, Claim 7 is easily seen to imply that

$$(i^1, o^1, \ldots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \ldots, j^L, p^L, i^{m'}, o^{m'}, \ldots, i^M, o^M).$$

is a CIC of μ for j^* at A that makes all agents $j' \le \hat{j}^*$ weakly better off than $(\mu + C) + \hat{C}$ and that weakly increases the welfare of i.

Case 1.2.4: $j^{\tilde{l}'} = j^*$. If $p^{\tilde{l}'} \in A_{j^*}$, the argument is analogous to that in Case 1.1.1.[15] Hence, we are left to consider the case of $j^{\tilde{l}'} = j^*$ and $p^{\tilde{l}'-1} \in \Omega_{j^*} \setminus A_{j^*}$. By analogs of our earlier arguments,

$$\tilde{C} \equiv (i^1, o^1, \ldots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \ldots, j^{\tilde{l}'-1}, p^{\tilde{l}'-1}).$$

is a CIC of μ for j^* at A that weakly increases the welfare of i. The definitions of $\tilde{m}$ and $\tilde{l}$ imply that $\tilde{m} \ge \underline{m}$. The first two parts of Claim 7 imply that if C increases the welfare of $i^{\tilde{m}}$ and $i^{\tilde{m}} < \hat{j}^*$, then $\tilde{m} = \underline{m}$ and $i^{\tilde{m}}$ is just as well off under $\mu + \tilde{C}$ as under $(\mu + C) + \hat{C}$. Since $\{p^{\tilde{l}'-1}, p^{\tilde{l}'}\} \subseteq \Omega_{j^*} \setminus A_{j^*}$, Claims 7 and 9 implies that there cannot be an $m > \underline{m}$ such that $i^m < \hat{j}^*$ and such that C increases the welfare of i^m. Combining the previous arguments, we find that $\tilde{C}$ is a CIC of μ for j^* at A that makes all agents $j' \le \hat{j}^*$ weakly better off then $(\mu + C) + \hat{C}$.

Case 2: $\hat{j}^*$ trades two undesirable objects in C. In this case, there exists an m' such that $i^{m'} = \hat{j}^*$, $o^{m'-1} \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$, and $o^{m'} \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$.

Assume first that $m(\underline{l}) > m'$ and consider the following sequence

$$\tilde{C} \equiv (i^1, o^1, \ldots, i^{m'-1}, o^{m'-1}, j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^M, o^M).$$

It is immediate that $\tilde{C}$ is a CIC of μ for j^* at A that weakly increases the welfare of i. Note that we must have $o^{m'-1} \ne p^1$ since $p^1 \in A_{\hat{j}^*}$ and, by assumption, $o^{m'-1} \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$.

[15] More precisely, let $\hat{l}$ be the first integer in $(\tilde{l}', \ldots, L, 1)$ such that either $j^{\hat{l}} \in I(C) \setminus \{j^*\}$ or $p^{\hat{l}} \in O(C)$. If $j^{\hat{l}} \in I(C) \setminus \{j^*\}$, let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. The definitions of $\tilde{l}'$, $\hat{l}$, and $\hat{m}$ imply $\hat{m} \ge m'$ and we can proceed just as in Case 1.1.1. If $j^{\hat{l}} \notin I(C) \setminus \{j^*\}$ and $p^{\hat{l}} \in O(C)$, let $\hat{m}$ be such that $o^{\hat{m}-1} = p^{\hat{l}}$. Since $j^{\hat{l}+1} = i^{\hat{m}-1}$, the definitions of $\tilde{l}'$, $\hat{l}$, and $\hat{m}$ imply that $\hat{m} \ge m'$ and we can again proceed as in Case 1.1.1.

Hence, $\hat{j}^*$ is indifferent between $\mu + \tilde{C}$ and $(\mu + C) + \hat{C}$. Claim 7 implies that there is no $m \in \{m' + 1, \ldots, \overline{m} - 1\}$ such that $i^m < \hat{j}^*$ and such that C increases the welfare of i^m. Thus, if $m(\underline{l}) \leq \overline{m}$, $\tilde{C}$ is a CIC for j^* at A such that all agents $j \leq \hat{j}^*$ are at least as well off under $\mu + \tilde{C}$ as under $(\mu + C) + \hat{C}$. If $m(\underline{l}) > \overline{m}$, then we must have $p^{\underline{l}} = o^{m(\underline{l})-1}$, $j^{\underline{l}+1} = i^{m(\underline{l})-1}$, and $m(\underline{l}) - 1 = \overline{m}$. If $i^{\overline{m}} < \hat{j}^*$, then, by the third part of Claim 7, $o^{\overline{m}} = p^{\underline{l}} \in A_{i^{\overline{m}}}$ implies $\{o^{\overline{m}-1}, p^{\underline{l}+1}\} \subseteq A_{i^{\overline{m}}}$. Hence, C cannot increase the welfare of $i^{\overline{m}}$ if $m(\underline{l}) > \overline{m}$. Therefore, $\tilde{C}$ is a CIC for j^* at A such that all agents $j \leq \hat{j}^*$ are at least as well off under $\mu + \tilde{C}$ as under $(\mu + C) + \hat{C}$.

Next, assume that $m(\underline{l}) = m'$. The definitions of $\underline{l}$ and $m(\underline{l})$ imply that we must have either $p^{\underline{l}} \in \mu(\hat{j}^*) \setminus (\mu + C)(\hat{j}^*)$ or that $\underline{l} = L, j^L \notin I(C)$, and $p^L \notin O(C)$. In the latter case, we obtain $O(C) \cap O(\hat{C}) = \emptyset$, which is impossible by Claim 4. In the former case, we have that $p^{\underline{l}} = o^{m'-1}$. However, given that $o \notin O(\hat{C})$ by Claim 6 and, by the assumption of Case 2, $o^{m'-1} = p^{\underline{l}} \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$,

$$\tilde{C}' \equiv (j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}})$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, which is impossible.

For the remainder of the proof in Case 2, we assume that $m(\underline{l}) < m'$. We divide our arguments into four subcases.

Case 2.1: $p^{\underline{l}} = o^M$. As in Case 1.1.3, let $\hat{l}$ be the first integer in $(\underline{l}+1, \ldots [L, 1] \ldots, \underline{l})$ such that $j^{\hat{l}} \in I(C)$.

If $j^{\hat{l}} \neq j^*$, we can proceed exactly as in the corresponding part of Case 1.1.3 since the argument there did not rely on the role of $\hat{j}^*$ in C.

For the remainder of the proof in Case 2.1., consider the sequence

$$\tilde{C} \equiv (i^1, o^1, \ldots, i^{m'-1}, o^{m'-1}, j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}).$$

Analogs of our earlier arguments show that $\tilde{C}$ is a CIC of μ for j^* at A that weakly increases the welfare of i. Hence, we must have $m^* \leq m' - 1$ and Claim 6 implies $m^* < \underline{m}$.

If $j^{\hat{l}} = j^*$ and $p^{\hat{l}} \in \Omega_{j^*} \setminus A_{j^*}$, Claims 7 and 9 imply that $\tilde{C}$ makes all agents $j' \leq \hat{j}^*$ weakly better off than $(\mu + C) + \hat{C}$.

If $j^{\hat{l}} = j^*$ and $p^{\hat{l}} \in A_{j^*}$, let $\hat{l}'$ be the first integer in $(\hat{l}, \ldots, L)$ such that either $j^{\hat{l}'} \in I(C) \setminus \{j^*\}$ or $p^{\hat{l}'} \in O(C) \cup \{p^L\}$.

If $j^{\hat{l}'} \in I(C) \setminus \{j^*\}$, let $\hat{m}'$ be such that $i^{\hat{m}'} = j^{\hat{l}'}$. Since $m^* < \underline{m} \leq \hat{m}'$, we obtain that

$$(j^{\hat{l}}, p^{\hat{l}}, \ldots, j^{\hat{l}'-1}, p^{\hat{l}'-1}, i^{\hat{m}'}, o^{\hat{m}'}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$, which is impossible.

If $j^{\hat{l}'} \notin I(C) \setminus \{j^*\}$ and $p^{\hat{l}'} \in O(C)$, let $\hat{m}'$ be such that $o^{\hat{m}'-1} = p^{\hat{l}'}$. Since $\hat{l}' \geq \hat{l}$, we must have $\hat{m}' - 1 \neq 1$. This implies that $\hat{m}' \geq \underline{m}$ and we proceed as in the case of $j^{\hat{l}'} \in I(C) \setminus \{j^*\}$.

If $j^{\hat{l}'} \notin I(C) \setminus \{j^*\}$ and $p^{\hat{l}'} \notin O(C)$, we have that $\hat{l}' = L$. Given that $m' \geq \underline{m}$, we proceed as in case of $j^{\hat{l}'} \in I(C) \setminus \{j^*\}$.

Case 2.2: $p^{\underline{l}} \neq o^M$ and $j^{\underline{l}+1} \neq j^*$. If $p^{\underline{l}} \in \mu(j^{\underline{l}+1}) \cap [(\mu + C)(j^{\underline{l}+1})]$, we have that $j^{\underline{l}+1} = i^{m(\underline{l})}$ and hence $m(\underline{l}) \geq \underline{m}$. By Claim 6, we must have $m^* \notin \{m(\underline{l}), \ldots, m' - 1\}$. But then, Claim 7 implies that

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{m'-1}, o^{m'-1})$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, which contradicts $\mu \in \mathcal{M}^{i-1}(\hat{A})$.

If $p^{\underline{l}} \in [(\mu + C)(j^{\underline{l}+1})] \setminus \mu(j^{\underline{l}+1})$, we must have $j^{\underline{l}+1} = i^{m(\underline{l})-1}$. If $m(\underline{l}) \geq \underline{m}$, we obtain a contradiction as in the case of $p^{\underline{l}} \in \mu(j^{\underline{l}+1}) \cap [(\mu + C)(j^{\underline{l}+1})]$. If $m(\underline{l}) < \underline{m}$, we must have $m(\underline{l}) = 2$ or $m(\underline{l}) = 1$. In the former case, we obtain a contradiction to our assumption that $j^{\underline{l}+1} \neq j^*$. In the latter case, we have that $j^{\underline{l}+1} = i^M$ and thus $p^{\underline{l}} = o^M$, which contradicts the premise of this subcase.

Case 2.3: $p^{\underline{l}} \in \Omega_{j^*} \setminus (A_{j^*} \cup \{o^M\})$ and $j^{\underline{l}+1} = j^*$. As in the case of $p^{\underline{l}} = o^M$, it follows that

$$(i^1, o^1, \ldots, i^{m'-1}, o^{m'-1}, j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}})$$

is a CIC of μ for j^* at A that makes all agents $j' \leq \hat{j}^*$ weakly better off then $(\mu + C) + \hat{C}$ and that weakly increases the welfare of i.

Case 2.4: $p^{\underline{l}} \in A_{j^*}$ and $j^{\underline{l}+1} = j^*$. Since $\hat{C}$ is a CIC for $\hat{j}^* > j^*$, we must have $p^{\underline{l}+1} \in A_{j^*}$ as well. Now define $\tilde{l}$ to be the first integer in $(\underline{l}+1, \ldots, L)$ such that either $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$ or $p^{\tilde{l}} \in O(C) \cup \{p^L\}$. Note that we allow for the case where $\tilde{l} = \underline{l} + 1$ and $p^{\underline{l}+1} \in O(C) \cup \{p^L\}$.

If $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$, let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$. We argue first that $m^* \geq \overline{m}$: otherwise, Claim 6 implies $m^* < \underline{m} \leq \tilde{m}$ and

$$(j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^{\tilde{l}-1}, p^{\tilde{l}-1}, i^{\tilde{m}}, o^{\tilde{m}}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$, thus contradicting $\mu \in \mathcal{M}^{i-1}(\hat{A})$. Now consider

$$\tilde{C} \equiv (j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{m'-1}, o^{m'-1}).$$

Since j^* trades two desirable objects in $\hat{C}$, Claim 8 implies that there cannot be an $m \leq m'-1$ such that $i^m < \hat{j}^*$ and such that C affects the welfare of i^m. Hence, given that $m^* \geq \overline{m} \geq m'$, we obtain that $\tilde{C}$ is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, thus contradicting $\mu \in \mathcal{M}^{i-1}(\hat{A})$.

Next, assume that $\tilde{l} < L$, $j^{\tilde{l}} \notin I(C) \setminus \{j^*\}$, and $p^{\tilde{l}} \in O(C)$. Let $\tilde{m}$ be such that $o^{\tilde{m}-1} = p^{\tilde{l}}$. Note that $\tilde{m} \geq \underline{m}$: if $\tilde{l} = L$, we must have $p^{\tilde{l}} = o^{m'}$ and hence $\tilde{m} > m' \geq \underline{m}$; if $\tilde{l} < L$, we have that $i^{\tilde{m}-1} = j^{\tilde{l}+1} \in I(C) \setminus \{j^*, \hat{j}^*\}$ and hence $\tilde{m} > \underline{m}$. Next, we argue that $m^* \geq \overline{m}$: otherwise, Claim 6 would imply that $m^* < \underline{m} \leq \tilde{m}$ and

$$(j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^{\tilde{l}}, p^{\tilde{l}}, i^{\tilde{m}}, o^{\tilde{m}}, \ldots, i^M, o^M)$$

would be a CIC of μ for j^* at $\hat{A}$, which is impossible. Now consider

$$\tilde{C}' \equiv (j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{m'-1}, o^{m'-1}).$$

Since $m^* \geq \overline{m} > m'-1$, an analogous argument to that in case $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$ shows that $\tilde{C}'$ is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, which is impossible.

Finally, if $\tilde{l} = L$, $j^L \notin I(C) \setminus \{j^*\}$, and $p^L \notin O(C)$, then

$$(j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^L, p^L, i^{m'}, o^{m'}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$, which is impossible, unless $m^* \geq m'$. However, by an analog of our earlier arguments, $m^* \geq m'$ implies that

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{m'-1}, o^{m'-1}).$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, which is another contradiction to $\mu \in \mathcal{M}^{i-1}(\hat{A})$.

Case 3: $\hat{j}^*$ exchanges two desirable objects in C. In this case, there exists an $m' > 1$ such that $\hat{j}^* = i^{m'}$ and $\{o^{m'-1}, o^{m'}\} \subseteq A_{\hat{j}^*}$.

First, $p^{\bar{l}} \notin O(C)$ and therefore $j^{\bar{l}} \in I(C) \setminus \{\hat{j}^*\}$: since $\hat{j}^*$ exchanges two desirable objects in C and since $p^L \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$, we must have $p^L \notin O(C)$; by Claim 5, $p^L \notin O(C)$ implies $p^{\bar{l}} \notin O(C)$.

If $m(\bar{l}) < m'$, then consider

$$\tilde{C} \equiv (i^1, o^1, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L, i^{m'}, o^{m'}, \ldots, i^M, o^M).$$

Note first, that $p^{\bar{l}} \notin O(C)$ implies $j^{\bar{l}} = i^{m(\bar{l})+1}$. Since $m(\bar{l}) < m'$, we have that $j^{\bar{l}} \neq j^*$ and $m(\bar{l}) + 1 \geq \underline{m}$. Since $m(\bar{l}) + 1 < m' \leq \overline{m}$, Claim 6 implies $m(\bar{l}) + 1 \neq m^*$. It is then easy to see that $\tilde{C}$ is a CIC of μ for j^* at A that weakly increases the welfare of i. Next, note that Claim 7 implies that there does not exist an $m \in \{\underline{m} + 1, \ldots, m' - 1\}$ such that $i^m < \hat{j}^*$ and such that C increases the welfare of i^m. Next, since $j^{\bar{l}} = j^*$ implies that $m(\bar{l}) = M$, we obtain a contradiction to our assumption that $m(\bar{l}) < m'$ unless $j^{\bar{l}} \neq j^*$. Since $j^{\bar{l}} = i^{m(\bar{l})+1}$, we have that $m(\bar{l}) + 1 \geq \underline{m}$. If $m(\bar{l}) + 1 = \underline{m}$ and $i^{\underline{m}} < \hat{j}^*$, then the second part of Claim 7 implies that C increases the welfare of $i^{\underline{m}}$ if and only if $\tilde{C}$ increases the welfare of $i^{\underline{m}}$. Combining the previous observations, we find that $\tilde{C}$ is a CIC of μ for j^* at A that provides all agents $j' \leq \hat{j}^*$ with the same welfare as $(\mu + C) + \hat{C}$.

Henceforth, assume that $m(\bar{l}) \geq m'$.

Next, we show that $j^{\bar{l}} = j^*$. Assume to the contrary that $j^{\bar{l}} \neq j^*$ and consider

$$\tilde{C}' \equiv (i^{m'}, o^{m'}, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L).$$

Since $i^{m(\bar{l})+1} = j^{\bar{l}} \neq j^*$, we have that $m(\bar{l}) + 1 \leq \overline{m}$. By the first part of Claim 7, there does not exist an $m \in \{m' + 1, \ldots, \overline{m} - 1\}$ such that $i^m < \hat{j}^*$ and such that C affects the welfare of i^m. If $m(\bar{l}) + 1 = \overline{m}$ and $i^{\overline{m}} < \hat{j}^*$, then the third part of Claim 7 implies that either $\{o^{m(\bar{l})}, p^{\bar{l}}\} \subseteq A_{j^{\bar{l}}}$ or $\{o^{m(\bar{l})}, p^{\bar{l}}\} \subseteq \Omega_{j^{\bar{l}}} \setminus A_{j^{\bar{l}}}$ so that $\tilde{C}'$ cannot affect the welfare of $i^{\overline{m}}$. Since $j^* \notin I(\tilde{C}')$, we obtain that $\tilde{C}'$ is a CIC of μ for $\hat{j}^*$ at A. Furthermore, given that $\underline{m} \leq m' < m(\bar{l}) + 1 \leq \overline{m}$, Claim 6 implies $m^* \notin \{m', \ldots, m(\bar{l})\}$. Hence, $\tilde{C}'$ is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, which contradicts $\mu \in \mathcal{M}^{i-1}(\hat{A})$.

Before proceeding, note that $j^{\bar{l}} = j^*$ implies $m(\bar{l}) = M$.

If $p^{\bar{l}} \in A_{j^*}$, then

$$\tilde{C}'' \equiv (i^{m'}, o^{m'}, \ldots, i^M, o^M, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L).$$

is a CIC of μ for j^* at A that weakly increases the welfare of i. Since $\{p^{\bar{l}-1}, p^{\bar{l}}\} \subseteq A_{j^*}$, Claims 7 and 8 imply that there does not exist an $m \in \{2, \ldots, m' - 1\}$ such that $i^m < \hat{j}^*$ and such that C increases the welfare of i^m. Hence, $\tilde{C}''$ provides all agents $j \leq \hat{j}^*$ with the same welfare as $(\mu + C) + \hat{C}$.

If $p^{\bar{l}-1} \in \Omega_{j^*} \setminus A_{j^*}$, we must have $p^{\bar{l}-1} \notin O(C)$.

We show first that $m^* \geq \overline{m}$. Consider the sequence

$$\tilde{C} \equiv (i^{m'}, o^{m'}, \ldots, i^M, o^M, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L).$$

Since $\{o^{m(\bar{l})}, p^{\bar{l}}\} \subseteq \Omega_{j^*} \setminus A_{j^*}$, $\tilde{C}$ does not affect the welfare of j^*. Furthermore, Claims 7 and 9 imply that there does not exist an $m \in \{m', \ldots, m(\bar{l})\}$ such that $i^m < \hat{j}^*$ and such that C increases the welfare of i^m. Combining our last two observations, if $m^* \notin \{m', \ldots, M\}$, then $\tilde{C}$ is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, which is impossible. Thus, $m^* \in \{m', \ldots, M\}$ and Claim 6 implies $m^* \geq \overline{m}$.

Now let $\tilde{l}$ be the last integer in $(1, \ldots, \bar{l}-1)$ such that $j^{\tilde{l}} \in I(C)$ and and let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$. Note that such an integer $\tilde{l}$ exists since $I(C) \cap I(\hat{C}) \neq \{j^*\}$ and $j^{\bar{l}} = j^*$. Since $m^* \geq \overline{m} > \tilde{m} - 1$,

$$(i^1, o^1, \ldots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \ldots, j^{\bar{l}-1}, p^{\bar{l}-1}).$$

is a CIC of μ for j^* at $\hat{A}$, which contradicts $\mu \in \mathcal{M}^{i-1}(\hat{A})$.

Case 4: $\hat{j}^* \notin I(C)$. Note first that $\hat{j}^* \notin I(C)$ implies $p^L \notin O(C)$. By Claim 5 we infer that $p^{\bar{l}} \notin O(C)$.

Assume first that $m(\bar{l}) < m(\underline{l})$.

Since $i^{m(\bar{l})+1} = j^{\bar{l}}$ and $m(\bar{l}) < m(\underline{l}) \leq M$, we must have $j^{\bar{l}} \neq j^*$. Therefore $m(\bar{l}) + 1 \geq \underline{m}$.

We first argue that it is impossible that $m(\bar{l}) + 1 = m(\underline{l})$ and $p^{\underline{l}} = o^{m(\underline{l})-1}$. Suppose to the contrary and note that we have $j^{\bar{l}} = i^{m(\underline{l})}$. Furthermore, given that $m(\underline{l}) \leq \overline{m}$, Claim 7 implies that if $i^{m(\underline{l})} < \hat{j}^*$, then either (A) $m(\underline{l}) \in \{\underline{m}+1, \ldots, \overline{m}\}$ and either $\{p^{\underline{l}}, p^{\bar{l}}\} \subseteq A_{i^{m(\underline{l})}}$ or $\{p^{\underline{l}}, p^{\bar{l}}\} \subseteq \Omega_{i^{m(\underline{l})}} \setminus A_{i^{m(\underline{l})}}$, or (B) $m(\underline{l}) = \underline{m}$ and $o^{\underline{m}} \in A_{i^{\underline{m}}}$ if and only if $p^{\bar{l}} \in A_{i^{\underline{m}}}$. But then

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L)$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, which is impossible.

Now consider

$$\tilde{C} \equiv (i^1, o^1, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots [p^L, j^1] \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^M, o^M).$$

Note first that $m(\bar{l}) + 1 < m(\underline{l})$ implies $\underline{m} \leq m(\bar{l}) + 1 < m(\underline{l}) - 1 \leq \overline{m}$. Hence, Claim 6 implies $m^* \neq m(\bar{l}) + 1$ and hence $i \neq j^{\bar{l}}$. It is then easy to see that $\tilde{C}$ is a CIC of μ for j^* at A that weakly increases the welfare of i.

Note that $\bar{l} > \underline{l}$ and hence $\hat{j}^*$ is just as well off under $\tilde{C}$ as under $(\mu + C) + \hat{C}$.

If $i^{m(\bar{l})+1} < \hat{j}^*$, then the first part of Claim 7 implies that C can increase the welfare of $i^{m(\bar{l})+1}$ only if $m(\bar{l}) + 1 = \underline{m}$. The second part of Claim 7 implies that $p^{\bar{l}} \in A_{i^{m(\bar{l})+1}}$ if and only if $o^{m(\bar{l})+1} \in A_{i^{m(\bar{l})+1}}$. Hence, if $i^{m(\bar{l})+1} = j^{\bar{l}} < \hat{j}^*$, then $i^{m(\bar{l})+1} = j^{\bar{l}}$ is just as well off under $\tilde{C}$ as under $(\mu + C) + \hat{C}$.

If $p^{\underline{l}} \notin O(C)$, we have that $i^{m(\underline{l})} = j^{\underline{l}+1}$ and hence $\underline{m} \leq m(\underline{l}) \leq \overline{m}$. By the first part of Claim 7, there is no $m \in \{m(\bar{l}) + 2, \ldots, m(\underline{l}) - 1\}$ such that $i^m < \hat{j}^*$ and such that C increases the welfare of i^m. If $i^{m(\underline{l})} < \hat{j}^*$ and C increases the welfare of $i^{m(\underline{l})}$, we must have $m(\underline{l}) = \overline{m}$. By the third part of Claim 7 we have that $o^{m(\underline{l})-1} \in \Omega_{i^{m(\underline{l})}} \setminus A_{i^{m(\underline{l})}}$ if and only if $p^{\underline{l}} \in \Omega_{i^{m(\underline{l})}} \setminus A_{i^{m(\underline{l})}}$. Hence, if $i^{m(\underline{l})} < \hat{j}^*$, $i^{m(\underline{l})}$ is at least as well off under $\mu + \tilde{C}$ as under $(\mu + C) + \hat{C}$. Hence, we have that $\tilde{C}$ is a CIC of μ for j^* at A that makes all agents $j' \leq \hat{j}^*$ at least as well off as $C + \hat{C}$.

If $p^{\underline{l}} \in O(C)$, we have that $p^{\underline{l}} = o^{m(\underline{l})-1}$. As established above, we must have $m(\underline{l}) > m(\bar{l}) + 1$ in this case. Hence, we obtain $m(\underline{l}) - 1 \geq \underline{m}$. Since $i^{m(\underline{l})-1} = p^{\underline{l}}$, we also have $m(\underline{l}) - 1 \leq \overline{m}$. Combining our last two observations, we proceed just as when $p^{\underline{l}} \notin O(C)$. This completes the proof in case $m(\underline{l}) > m(\bar{l})$.

For the remainder of the proof in Case 4, assume that $m(\underline{l}) \leq m(\bar{l})$. We distinguish two subcases.

Case 4.1: $\boldsymbol{m(\bar{l}) = M}$ In this case $j^{\bar{l}} = j^*$.

If $p^{\bar{l}} \in A_{j^*}$, then Claims 7 and 9 imply that there cannot exist an $m \in \{2, \ldots, \underline{m}\}$ such that $i^m < \hat{j}^*$ and such that C affects the welfare of i^m. Hence,

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^M, o^M, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L)$$

is a CIC of μ for j^* at A such that all agents $j \leq \hat{j}^*$ are at least as well off as under $(\mu + C) + \hat{C}$. Furthermore, the definitions of $\underline{l}$ and $\bar{l}$ together with our assumption that $j^{\bar{l}} = j^*$ imply $i \notin \{j^1, \ldots, j^{\underline{l}}, j^{\bar{l}}, \ldots, j^L\}$. Since i receives the desirable object o in C, it is then easy to see that the above CIC for j^* weakly increases the welfare of i.

If $p^{\bar{l}} \in \Omega_{j^*} \setminus A_{j^*}$, then $p^{\bar{l}-1} \in \Omega_{j^*} \setminus A_{j^*}$ and $p^{\bar{l}-1} \notin O(C)$.

We argue first that $m^* < \underline{m}$ in this case. Let $\tilde{l}$ be the last integer in $(1, \ldots, \bar{l} - 1)$ such that $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$ and let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$. Consider

$$\tilde{C}' \equiv (i^1, o^1, \ldots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \ldots, j^{\bar{l}-1}, p^{\bar{l}-1}).$$

If $m^* > \tilde{m}-1$, $\tilde{C}'$ is a CIC of μ for j^* at $\hat{A}$, which is impossible. Thus, $m^* < \tilde{m}-1$. Since $\tilde{m} \geq \underline{m}$, Claim 6 implies that $m^* < \underline{m}$.

Next, we show that $\underline{l} < \bar{l}-1$: otherwise, $\underline{l} = \bar{l}-1$ and, given that $p^{\bar{l}-1} \notin O(C)$ we have $\{p^1, \ldots, p^{\bar{l}-1}\} \cap O(C) = \emptyset$; given that $p^L \notin O(C)$, Claim 5 and the definition of $\bar{l}$ imply $\{p^{\bar{l}}, \ldots, p^L\} \cap O(C) = \emptyset$, which contradicts Claim 4.

As a next step, we argue that $\underline{l} < \bar{l}-1$ implies $m(\underline{l}) \geq \underline{m}$: if $p^{\underline{l}} \notin O(C)$, we have $j^{\underline{l}+1} = i^{m(\underline{l})}$ and $\underline{l} < \bar{l}-1$ implies $j^{\underline{l}+1} \neq j^*$; if $p^{\underline{l}} \in O(C)$, we have $j^{\underline{l}+1} = i^{m(\underline{l})-1}$ and $\underline{l} < \bar{l}-1$ again implies $j^{\underline{l}+1} \neq j^*$.

If $o^{m(\bar{l})} \neq p^{\bar{l}}$, Claims 7 and 9 as well as $m^* < \underline{m}$ are easily seen to imply that

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L)$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, which is impossible.

If $o^{m(\bar{l})} = p^{\bar{l}}$, we obtain a contradiction to Claim 5 since $p^{\bar{l}} \in O(C)$ implies $\bar{l} = L$ but we have $p^L \notin O(C)$ given that $\hat{j}^* \notin I(C)$.

Case 4.2: $\boldsymbol{m(\bar{l}) < M}$ Note that $j^{\bar{l}} \neq j^*$ in this case. Since $i^{m(\bar{l})+1} = j^{\bar{l}}$ and $m(\bar{l})+1 \in \{2, \ldots, M\}$, we obtain $m(\bar{l}) < \overline{m}$.

Next, we argue that $m(\underline{l}) < \underline{m}$. Assume to the contrary that $m(\underline{l}) \geq \underline{m}$ and consider the sequence

$$\tilde{C}'' \equiv (j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L).$$

Since $\underline{m} \leq m(\underline{l}) \leq m(\bar{l}) < \overline{m}$, Claim 6 implies $m^* \notin \{m(\underline{l}), \ldots, m(\bar{l})\}$. From Claim 7 we infer that there is no $m \in \{m(\underline{l})+1, \ldots, m(\bar{l})\}$ such that $i^m < \hat{j}^*$ and such that C decreases the welfare of i^m. If $m(\underline{l}) = \underline{m}$ and $i^{\underline{m}} < \hat{j}^*$, then the second part of Claim 7 implies that either $\{p^{\underline{l}}, o^{m(\underline{l})}\} \subseteq A_{i^{\underline{m}}}$ or $\{p^{\underline{l}}, o^{m(\underline{l})}\} \subseteq \Omega_{i^{\underline{m}}} \setminus A_{i^{\underline{m}}}$. Hence, $\tilde{C}''$ can also not affect the welfare of $i^{m(\underline{l})}$ if $i^{m(\underline{l})} < \hat{j}^*$. Summing up our previous observations, we obtain that $\tilde{C}''$ is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, which contradicts $\mu \in \mathcal{M}^{i-1}(\hat{A})$.

Now note that $m(\underline{l}) < \underline{m}$ is possible only if either $m(\underline{l}) = 1$ or $i^{m(\underline{l})} \notin I(\hat{C})$ and $m(\underline{l}) = 2$. We split the remainder of the proof into four subcases.

Case 4.2.1: $\boldsymbol{m(\underline{l}) = 1}$ and $\boldsymbol{p^{\underline{l}} \in O(C)}$. In this subcase, $p^{\underline{l}} = o^M$ and

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^1, o^1, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L)$$

is a CIC of μ for j^* at A. Furthermore, given that j^* exchanges two undesirable objects in $\hat{C}$, analogs of our earlier arguments show that, by Claims 7 and 9, all agents $j \le \hat{j}^*$ are at least as well off as under $(\mu + C) + \hat{C}$. If $j^{\bar{l}} = i$, the above sequence would be a CIC of μ for j^* at $\hat{A}$ given that i would not receive o in this CIC. In any other case, the CIC weakly increases the welfare of i.

Case 4.2.2: $\boldsymbol{m(\underline{l}) = 1}$ and $\boldsymbol{p^{\underline{l}} \in \Omega_{j^*} \setminus (A_{j^*} \cup O(C))}$. Analogs of our earlier arguments show that

$$(i^1, o^1, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots [p^L, j^1] \ldots, j^{\underline{l}}, p^{\underline{l}})$$

is a CIC of μ for j^* at A with the desired properties.

Case 4.2.3: $\boldsymbol{m(\underline{l}) = 1}$ and $\boldsymbol{p^{\underline{l}+1} \in A_{j^*}}$. Let $\tilde{l}$ be the first integer in $(\underline{l}+1, \ldots, \bar{l})$ such that either $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$ or $p^{\tilde{l}} \in O(C)$. Note that an $\tilde{l}$ with the desired properties has to exist since $j^{\bar{l}} \neq j^*$ and $\bar{l} > \underline{l} + 1$.

If $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$, let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$ and consider

$$(j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^{\tilde{l}-1}, p^{\tilde{l}-1}, i^{\tilde{m}}, o^{\tilde{m}}, \ldots, i^M, o^M).$$

Unless $m^* \ge \overline{m} \ge m(\bar{l}) + 1$, the preceding sequence is a CIC of μ for j^* at $\hat{A}$, which is impossible. However, if $m^* \ge \overline{m}$, then

$$\tilde{C}''' \equiv (j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L).$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$ given that Claims 7 and 8 imply that $\tilde{C}'''$ cannot affect the welfare of any agent $j < \hat{j}^*$. This contradicts the assumption that $\mu \in \mathcal{M}^{i-1}(\hat{A})$.

If $j^{\tilde{l}} \notin I(C) \setminus \{j^*\}$, let $\tilde{m}$ be such that $o^{\tilde{m}-1} = p^{\tilde{l}}$ and consider

$$(j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^{\tilde{l}}, p^{\tilde{l}}, i^{\tilde{m}}, o^{\tilde{m}}, \ldots, i^M, o^M).$$

We must have $j^{\tilde{l}+1} = i^{\tilde{m}-1}$. Since $\tilde{l} + 1 \in \{\underline{l} + 2, \ldots, \bar{l}\}$, we must have $j^{\tilde{l}+1} \neq j^*$ and therefore $\tilde{m} - 1 \ge \underline{m}$. The remainder of the argument is exactly as when $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$.

Case 4.2.4: $\boldsymbol{m(\underline{l}) = 2}$ and $\boldsymbol{i^{m(\underline{l})} \notin I(\hat{C})}$. The premise of this subcase implies that $p^{\underline{l}} = o^1$. Hence, we must have $j^{\underline{l}+1} = j^*$ as well as $p^{\underline{l}+1} \in A_{j^*}$ (given that $p^{\underline{l}} = o^1 \in A_{j^*}$) and we proceed as in Case 4.2.3.

□

Proof of Theorem 3. Let A be a profile of desirable sets and, for some $i \in I$, let $\hat{A}_i$ be an alternative report for i.

We argue first that it is without loss of generality to assume that $\hat{A}_i \setminus A_i = \emptyset$. To see this, assume that $\hat{A}_i \setminus A_i \neq \emptyset$ and let $o \in \hat{A}_i \setminus A_i$. First suppose that $o \notin \Omega_i \cup A_i$. If $K^i(\hat{A}_i \setminus \{o\}) = K^i(\hat{A}_i)$, then it is immediate that we can consider the manipulation $\hat{A}_i \setminus \{o\}$ rather than $\hat{A}_i$. If $K^i(\hat{A}_i \setminus \{o\}) \neq K^i(\hat{A}_i)$, then Lemma 1 implies that $o \in \mu(i)$ for all $\mu \in \mathcal{M}^i(\hat{A})$. Since $o \in \hat{A}_i \setminus (\Omega_i \cup A_i)$, each matching in $\mathcal{M}^i(\hat{A})$ is unacceptable to i. Now suppose that $o \in \Omega_i$. Then by Lemma 2, for each $\mu \in \mathcal{M}^i(\hat{A})$ and each $\mu' \in \mathcal{M}^i(A)$, $|\mu'(i) \cap A_i| = K^i(A) \geq |\mu(i) \cap A_i|$. Hence, reporting $\hat{A}'_i \equiv \hat{A}_i \setminus \{o\}$ is at least as good for i as $\hat{A}_i$. Applying these arguments $|\hat{A}_i \setminus A_i|$ times, we end up with a manipulation $\hat{A}'_i$ that does at least as well as $\hat{A}_i$ and that satisfies $\hat{A}'_i \subseteq A_i$.

For the remainder of the proof we assume that $\hat{A}_i \setminus A_i = \emptyset$ and $A_i \setminus \hat{A}_i \neq \emptyset$. Let $\{o^1, \dots, o^M\} = A_i \setminus \hat{A}_i$ for some $M \geq 1$. By Lemma 3 (taking $\hat{A}_i \cup \{o^1\}$ to be the true and $\hat{A}_i$ to be the false set of desirable objects), we obtain $K^i(\hat{A}_i \cup \{o^1\}) \geq K^i(\hat{A}_i)$. Proceeding inductively, M applications of Lemma 3 yield that $K^i(\hat{A}_i \cup \{o^1, \dots, o^M\}) \geq K^i(\hat{A}_i)$. Hence, by definition of $o^1, \dots, o^M$ we obtain that $K^i(A) \geq K^i(\hat{A}_i)$. This completes the proof.

□

D Proofs from Section 5

Proof of Theorem 4.

1. We prove, by induction on t that the maximum flows of (V, E, q) correspond to $\mathcal{M}^t(A)$ after the tth iteration of the for-loop on line 2 of Algorithm 1.

   As a base case, we show that for (V, E, q), where q is the initial capacity vector defined above, f is a maximum flow if and only if it corresponds to some matching $\mu \in \mathcal{M}^0(A)$. Take any matching $\mu \in \mathcal{M}^0(A)$. The definition of a matching and trichotomous preferences imply $|\mu(i)| = |\Omega_i|$ and $\mu(i) \subseteq \Omega_i \cup A_i$. Define a flow f by setting, for all $i \in I$, $f(S, i) = |\Omega_i|$, $f(i, i^A) = |\mu(i) \cap A_i|$, $f(i, i^U) = |\mu(i) \cap (\Omega_i \setminus A_i)|$, $f(i^A, o) = 1$ for all $o \in \mu(i) \cap A_i$, $f(i^U, o) = 1$ for all $o \in \mu(i) \cap (\Omega_i \setminus A_i)$, and, for all $o \in \cup_{i \in I} \mu(i)$, $f(o, T) = 1$. Clearly, f is a maximum flow in (V, E, q). Conversely, take a maximum flow f in (V, E, q) and let μ be the matching that corresponds to f. Since all maximum flows must have value $\sum_i |\Omega_i|$, we must have $|\mu(i)| = |\Omega_i|$ for all $i \in I$. Since (V, E, q) contains no edges between i and an object in $O \setminus (\Omega_i \cup A_i)$, we have $\mu(i) \subseteq \Omega_i \cup A_i$. Hence, $\mu \in \mathcal{M}^0(A)$.

As an induction hypothesis, suppose that at the start of the tth iteration of the loop, the maximum flows of (V, E, q) correspond to $\mathcal{M}^{t-1}(A)$. Let $\hat{q}$ be the capacity at the end of the tth iteration of the for-loop, f be a maximum flow in $(V, E, \hat{q})$, and μ be the matching that corresponds to f. We show first that $\mu \in \mathcal{M}^t(A)$. By construction, $|\mu(t) \cap A_t| = |\Omega_t| - \hat{q}(i, i^U)$. Assume that there is a matching $\mu' \in \mathcal{M}^t(A)$ such that $|\mu'(t) \cap A_t| > |\mu(t) \cap A_t|$. Since the set of maximum flows of (V, E, q) corresponds to $\mathcal{M}^{t-1(A)}$ and since $\mu' \in \mathcal{M}^t(A)$, just as argued in the base case, μ' induces a maximum flow f' in (V, E, q). Given that $|\mu'(t) \cap A_t| > |\mu(t) \cap A_t|$, f' is a maximum flow in $(V, E, \tilde{q})$, where $\tilde{q}(t, t^U) = \hat{q}(t, t^U) - 1$ and $\tilde{q}(e) = \hat{q}(e)$ for all $e \in E \setminus \{(t, t^U)\}$. But then, the tth iteration of the loop would have concluded with the capacity of (i, i^U) no higher than $\hat{q}(t, t^U) - 1$. This contradiction shows that $\mu \in \mathcal{M}^t(A)$. Conversely, take any matching $\hat{\mu} \in \mathcal{M}^t$ and let $\hat{f}$ be the induced flow, defined as in the base case. By our previous arguments, we must have $|\hat{\mu}(t) \cap A_t| = |\mu(t) \cap A_t|$. Since $|\hat{\mu}(t)| = |\mu(t)| = |\Omega_t|$, we obtain that $|\hat{\mu}(t) \setminus A_t| = |\mu(t) \setminus A_t| = \hat{q}(t, t^U)$. Given that $\mu \in \mathcal{M}^t \subseteq \mathcal{M}^{t-1}$ and that $\hat{q}(e) = q(e)$ for all $e \in E \setminus \{(t, t^U)\}$, our inductive assumption then implies that $\hat{f}$ is a maximum flow in $(V, E, \hat{q})$.

2. Let $m = |O|$ and $n = |I|$. Then there are at most m edges out of i^A for each $i \in I$, so the Ford-Fulkerson algorithm computes MaxFlow in $O(mn^2)$ time. We invoke MaxFlow while minimizing the flow along (i, i^U), so it suffices to call it $O(log(k))$ times for each agent, where $k = \max_{i \in N} |\Omega_i|$. Thus, the complexity of IRP is $O(mn^3 log(k))$. Fixing Ω_i for each $i \in N$ rather than taking it as a parameter, the complexity is $O(mn^3)$ since k is a constant.

□

Strategy-Proof Exchange under Trichotomous Preferences

Online Appendix

October 19, 2018

A Claims in Proof of Lemma 3

Proof of Claim 1. Since $\mu \in \mathcal{M}^{i-1}(\hat{A})$ and $j^* < i$, we obtain $\mu(j^*) \cap \hat{A}_{j^*} = K^{j^*}(\hat{A})$. Since $K^{j^*}(A) > K^{j^*}(\hat{A})$, we obtain $\mu \notin \mathcal{M}^{j^*}(A)$.

On the other hand, $j^* < i$ also implies that $\mu \in \mathcal{M}^{j^*-1}(\hat{A})$ given that $\mathcal{M}^{i-1}(\hat{A}) \subseteq \mathcal{M}^{j^*-1}(\hat{A})$. Since $\hat{A}_j \subseteq A_j$ for all $j \in I$, the definition of j^* implies that $\mathcal{M}^{j^*-1}(\hat{A}) \subseteq \mathcal{M}^{j^*-1}(A)$ and thus $\mu \in \mathcal{M}^{j^*-1}(A)$. $\square$

Proof of Claim 2. If not, then $\mu + C \in \mathcal{M}^{j^*-1}(\hat{A})$ since $\mu \in \mathcal{M}^{j^*-1}(\hat{A})$, $o \notin (\mu + C)(i)$, $\hat{A}_i = A_i \setminus \{o\}$, and $\hat{A}_j = A_j$ for all $j \neq i$. But $\mu + C \in \mathcal{M}^{j^*-1}(\hat{A})$ contradicts $\mu \in \mathcal{M}^{i-1}(\hat{A}) \subseteq \mathcal{M}^{j^*}(\hat{A})$ since $|(\mu + C)(j^*) \cap \hat{A}_{j^*}| > |\mu(j^*) \cap \hat{A}_{j^*}|$. $\square$

Proof of Claim 3. Note that the first part of Theorem 2 implies that we must have $\hat{j}^* \geq j^*$ since $\mu + C \in \mathcal{M}^{j^*-1}(A)$ given that $\mu \in \mathcal{M}^{j^*-1}(A)$ and that C is a CIC of μ for j^* at A. Given that $\hat{j}^* \geq j^*$ and $\mu \in \mathcal{M}^{j^*-1}(A)$, our assumption that $\hat{j}^*$ is the highest priority agent for whom there exists a CIC of $\mu + C$ at A and the second part of Theorem 2 imply $\mu + C \in \mathcal{M}^{\hat{j}^*-1}(A)$. Furthermore, we must also have $(\mu + C) + \hat{C} \in \mathcal{M}^{\hat{j}^*-1}(A)$ since $\mu + C \in \mathcal{M}^{\hat{j}^*-1}(A)$ and since $\hat{C}$ is a CIC of $\mu + C$ for $\hat{j}^*$ at A. $\square$

Proof of Claim 4. If $O(C) \cap O(\hat{C}) = \emptyset$, then by Claim 2, $o \notin O(\hat{C})$. Furthermore, $O(C) \cap O(\hat{C}) = \emptyset$ also implies that for all j and all $o' \in O(\hat{C})$, $o' \in (\mu + C)(j)$ if and only if $o' \in \mu(j)$. Combining the last two observations, we find that $\hat{C}$ is a CIC of μ for $\hat{j}^*$ at $\hat{A}$. Since $\hat{j}^* < i$, we contradict our assumption that $\mu \in \mathcal{M}^{i-1}(\hat{A}) \subseteq \mathcal{M}^{\hat{j}^*}(\hat{A})$. $\square$

We now introduce one additional claim that is used to prove the remaining claims from the main body of the paper. The claim states that C and $\hat{C}$ treat agents with higher priority than their respective heads, j^* and $\hat{j}^*$, "consistently."

Lemma 8. *For each pair m and l such that $i^m = j^l \equiv \underline{j}$ and $\underline{j} < \min\{j^*, \hat{j}^*\}$, either*

$$\{o^{m-1}, o^m, p^{l-1}, p^l\} \subseteq A_{\underline{j}} \text{ or } \{o^{m-1}, o^m, p^{l-1}, p^l\} \subseteq \Omega_{\underline{j}} \setminus A_{\underline{j}}.$$

Proof of Lemma 8. Assume to the contrary that there is an agent $\underline{j} < \min\{j^*, \hat{j}^*\}$ for whom the statement of the claim is violated. Without loss of generality suppose that there is no agent with higher priority than $\underline{j}$ for whom Lemma 8 does not hold, that is, for all pairs m', l' such that $i^{m'} = j^{l'}$ and $i^{m'} < \underline{j}$, either

$$\{o^{m'-1}, o^{m'}, p^{l'-1}, p^{l'}\} \subseteq A_{i^{m'}} \text{ or } \{o^{m'-1}, o^{m'}, p^{l'-1}, p^{l'}\} \subseteq \Omega_{i^{m'}} \setminus A_{i^{m'}}.$$

Since C and $\hat{C}$ are CICs for agents with lower priority than $\underline{j}$, there are only two cases to consider. In either case, we construct a CIC of μ for $\underline{j}$ at A. Since $\underline{j} < \min\{j^*, \hat{j}^*\}$, Theorem 2 and Claim 1 imply that the existence of such a CIC contradicts our assumption that $\mu \in \mathcal{M}^{i-1}(\hat{A})$.

Case 1: $\boldsymbol{\{o^{m-1}, o^m\} \subseteq \Omega_{\underline{j}} \setminus A_{\underline{j}}}$ and $\boldsymbol{\{p^{l-1}, p^l\} \subseteq A_{\underline{j}}}$. Let $\tilde{l}$ be the first integer in $(l, \ldots [L, 1] \ldots, l-1)$ such that either $j^{\tilde{l}} \in I(C) \setminus \{\underline{j}\}$ or $p^{\tilde{l}} \in O(C)$. Note that the existence of an $\tilde{l}$ with the desired properties follows since $O(C) \cap O(\hat{C}) \neq \emptyset$ by Claim 4. If $j^{\tilde{l}} \notin I(C) \setminus \{\underline{j}\}$, so that $p^{\tilde{l}} \in O(C)$, let $\hat{l} = \tilde{l}$ and let $\hat{m}$ be such that $o^{\hat{m}-1} = p^{\hat{l}}$. If $j^{\tilde{l}} \in I(C) \setminus \{\underline{j}\}$, let $\hat{l} = \tilde{l} - 1$ and let $\hat{m}$ be such that $i^{\hat{m}} = j^{\tilde{l}} = j^{\hat{l}+1}$.

Assume first that $\hat{m} = m$ and consider

$$\tilde{C} \equiv (j^l, p^l, \ldots [p^L, j^1] \ldots, j^{\hat{l}}, p^{\hat{l}}).$$

In this case, we must have $j^{\tilde{l}} \notin I(C) \setminus \{\underline{j}\}$ since otherwise $i^{\hat{m}} = j^{\tilde{l}}$, which is impossible given that $i^{\hat{m}} = i^m = \underline{j}$. Hence, $\tilde{l} = \hat{l}$ and $p^{\hat{l}} \in O(C)$. Since $\hat{m} = m$, we obtain $p^{\hat{l}} = o^{m-1}$. Since $o^{m-1} \in \Omega_{\underline{j}} \setminus A_{\underline{j}}$ and $p^l \in A_{\underline{j}}$, we obtain $\hat{l} \neq l$. Furthermore, the definition of $\hat{l}$ implies that $\{j^l, p^l, \ldots [p^L, j^1] \ldots, j^{\hat{l}}, p^{\hat{l}}\} \cap \{i^1, o^1, \ldots, i^M, o^M\} = \{p^{\hat{l}}\}$. Since C is a CIC of μ, we have that $p^{\hat{l}} = o^{m-1} \in \mu(i^m) = \mu(\underline{j})$. Combining the last two observations with the fact that $\hat{C}$ is a CIC of $\mu + C$ for $\hat{j}^* > \underline{j}$, we obtain that $\tilde{C}$ is a CIC of μ for $\underline{j}$ at $\hat{A}$. Since $\underline{j} < j^* \leq i-1$, we obtain a contradiction to our assumption that $\mu \in \mathcal{M}^{i-1}(\hat{A})$.

Henceforth, we assume that $\hat{m} \neq m$.

We claim that

$$\tilde{C} \equiv (j^l, p^l, \ldots [p^L, j^1] \ldots, j^{\hat{l}}, p^{\hat{l}}, i^{\hat{m}}, o^{\hat{m}}, \ldots [o^M, i^1] \ldots, i^{m-1}, o^{m-1})$$

is a CIC of μ for $\underline{j}$ at A.[16]

We argue first, that the objects and agents in $\tilde{C}$ are distinct. Since $\hat{m} \neq m$ and since C

[16]See Figure 2 for an illustration.

is a CIC, we have that $j^l = i^m \notin \{i^{\hat{m}}, \dots [i^M, i^1] \dots, i^{m-1}\}$. Given that our definition of $\hat{l}$ implies $j^{\hat{l}} \in I(C)$ only if $j^{\hat{l}} = j^l = \underline{j}$, the observation in the previous sentence implies $\{j^l, p^l, \dots [p^L, j^1] \dots, j^{\hat{l}}, p^{\hat{l}}\} \cap \{i^{\hat{m}}, o^{\hat{m}}, \dots [o^M, i^1] \dots, i^{m-1}, o^{m-1}\} \subseteq \{p^{\hat{l}}\}$. If $p^{\hat{l}} \in O(C)$, then we must have $p^{\hat{l}} = o^{\hat{m}-1}$. However, since $\hat{m} - 1 \notin \{\hat{m}, \dots [M, 1] \dots, m-1\}$ given that $\hat{m} \neq m$ and since C is a CIC, we obtain that $p^{\hat{l}} \notin \{o^{\hat{m}}, \dots [o^M, o^1] \dots, o^{m-1}\}$.

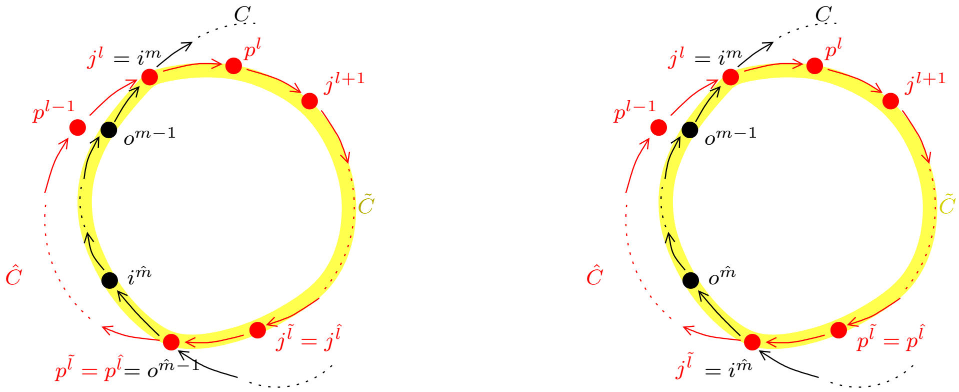
Second, we argue that $\tilde{C}$ is a cycle of μ. For each $k \in \{l+1, \dots [L, 1] \dots, \hat{l}\}$ such that $j^k \neq \underline{j}$, the definition of $\hat{l}$ implies that we must have $j^k \notin I(C)$ and hence $(\mu + C)(j^k) = \mu(j^k)$. Since $\hat{C}$ is a CIC of $\mu + C$, $p^{k-1} \in (\mu + C)(j^k) = \mu(j^k)$ and $p^k \notin (\mu + C)(j^k) = \mu(j^k)$. For each $k \in \{\hat{m}, \dots [M, 1] \dots, m\}$, since C is a CIC of μ, $o^{k-1} \in \mu(i^k)$ and $o^k \notin \mu(i^k)$. Since $i^m = j^l = \underline{j}$ and $o^{m-1} \in \mu(i^m)$, $o^{m-1} \in \mu(j^l)$. Finally, we check that $p^{\hat{l}} \in \mu(i^{\hat{m}})$: if $j^{\tilde{l}} \notin I(C) \setminus \{\underline{j}\}$, then $p^{\hat{l}} = o^{\hat{m}-1}$ and since C is a CIC of μ, $o^{\hat{m}-1} \in \mu(i^{\hat{m}})$; if $j^{\tilde{l}} \in I(C) \setminus \{\underline{j}\}$, then $i^{\hat{m}} = j^{\tilde{l}}$ and $p^{\hat{l}} = p^{\tilde{l}-1} \notin O(C)$, so that $p^{\hat{l}} \in \mu(i^{\hat{m}})$ since $\hat{C}$ is a CIC of $\mu + C$ and $p^{\tilde{l}-1} \in (\mu + C)(j^{\tilde{l}})$.

Third, we show that $\tilde{C}$ is individually rational at A. Since $\hat{C}$ is a CIC, for each $k \in \{l, \dots [L, 1] \dots, \hat{l}\}$, $p^k \in \Omega_{j^k} \cup A_{j^k}$. Since C is a CIC, for each $k \in \{\hat{m}, \dots [M, 1] \dots, m-1\}$, $o^k \in \Omega_{i^k} \cup A_{i^k}$.

Finally, we argue that $\tilde{C}$ is a CIC for $\underline{j}$. Since $j^l = \underline{j}$, $o^{m-1} \in \Omega_{\underline{j}} \setminus A_{\underline{j}}$, $p^l \in A_{\underline{j}}$, and since all agents in $I(\tilde{C})$ are distinct, $\tilde{C}$ improves the welfare of $\underline{j}$. Hence, it remains to be shown that $\tilde{C}$ does not affect any agent with higher priority than $\underline{j}$. Note that, for all $k \in \{l+1, \dots [L, 1] \dots, \hat{l}\}$ such that $j^k < \underline{j}$, we must have either $\{p^{k-1}, p^k\} \subseteq A_{j^k}$ or $\{p^{k-1}, p^k\} \subseteq \Omega_{j^k} \setminus A_{j^k}$ given that $\hat{C}$ is a CIC for $\hat{j}^*$ and that $\underline{j} < \hat{j}^*$. Similarly, for all $k \in \{\hat{m}+1, \dots [M, 1] \dots, m-1\}$ such that $i^k < \underline{j}$, either $\{o^{k-1}, o^k\} \subseteq A_{i^k}$ or $\{o^{k-1}, o^k\} \subseteq \Omega_{i^k} \setminus A_{i^k}$ given that C is a CIC for j^* and that $\underline{j} < j^*$. Finally, consider $i^{\hat{m}}$ and assume that $i^{\hat{m}} < \underline{j}$. By the choice of $\underline{j}$ as the highest priority agent for whom the statement of Lemma 8 is violated, either $\{p^{\hat{l}}, o^{\hat{m}}\} \subseteq A_{i^{\hat{m}}}$ or $\{p^{\hat{l}}, o^{\hat{m}}\} \subseteq \Omega_{i^{\hat{m}}} \setminus A_{i^{\hat{m}}}$. Hence, $\hat{C}$ does not affect $i^{\hat{m}}$ and this completes the proof.

Case 2: $\{o^{m-1}, o^m\} \subseteq A_{\underline{j}}$ and $\{p^{l-1}, p^l\} \subseteq \Omega_{\underline{j}} \setminus A_{\underline{j}}$. Let $\tilde{l}$ be the last integer in $(l, \dots [L, 1] \dots, l-1)$ such that either $j^{\tilde{l}} \in I(C) \setminus \{\underline{j}\}$ or $p^{\tilde{l}} \in O(C)$. Claim 4 ensures that such $\tilde{l}$ exists.

We now establish that $p^{\tilde{l}} \notin O(C)$. If there is $\tilde{m}$ such that $p^{\tilde{l}} = o^{\tilde{m}}$, then since $p^{\tilde{l}} \in (\mu + C)(j^{\tilde{l}+1})$, we have $j^{\tilde{l}+1} = i^{\tilde{m}} \in I(C)$. Since $j^{\tilde{l}+1} \in I(C)$ is a contradiction to the definition of $\tilde{l}$ if $\tilde{l} \neq l-1$, we must have $\tilde{l} = l-1$. Given that $p^{l-1} \in (\mu + C)(\underline{j})$ and C is a CIC of μ, $p^{l-1} \in O(C)$ is possible only if $p^{l-1} = o^m$. However, since $o^m \in A_{\underline{j}}$ and $p^{l-1} \in \Omega_{\underline{j}} \setminus A_{\underline{j}}$, we have $o^m \neq p^{l-1}$ and hence, by our previous arguments, $p^{l-1} \notin O(C)$.



(a) Case $j^{\tilde{l}} \notin I(C) \setminus \{\underline{j}\}$ (b) Case $j^{\tilde{l}} \in I(C) \setminus \{\underline{j}\}$

Figure 2: Construction of $\tilde{C}$ for Case 1 in the proof of Claim 8.

Since $p^{\tilde{l}} \notin O(C)$, we have $j^{\tilde{l}} \in I(C)$ so that there is an $\hat{m}$ be such that $i^{\hat{m}+1} = j^{\tilde{l}}$.

By an argument analogous to that in Case 1, the following is a CIC of μ for $\underline{j}$ at A:[17]

$$\tilde{C} \equiv (i^m, o^m, \ldots [o^M, i^1] \ldots, i^{\hat{m}}, o^{\hat{m}}, j^{\tilde{l}}, p^{\tilde{l}}, \ldots [p^L, j^1] \ldots, j^{l-1}, p^{l-1})$$

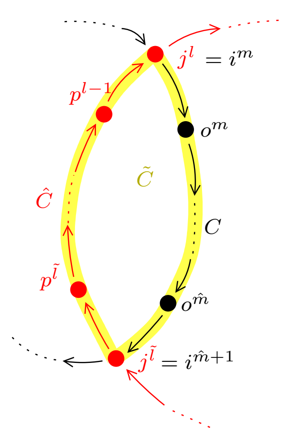


Figure 3: Construction of $\tilde{C}$ for Case 2 in the proof of Lemma 8.

□

Proof of Claim 5. Suppose that $p^{\bar{l}} \in O(C)$. Since $\hat{C}$ is a CIC of $\mu + C$, we have $j^{\bar{l}+1} \in I(C)$. However, if $\bar{l} < L$, then $\bar{l} + 1 \neq 1$, so $j^{\bar{l}+1} \neq j^1 = \hat{j}^*$. This contradicts the definition of $\bar{l}$. □

[17] See Figure 3 for an illustration.

Proof of Claim 6. We argue first that if $\underline{m} \leq m^*$, then $o \notin O(\hat{C})$ and $\overline{m} \leq m^*$.

Assume to the contrary that $\underline{m} \leq m^*$ and either $o \in O(\hat{C})$ or $m^* < \overline{m}$. Let $\tilde{m}_1, \ldots, \tilde{m}_T$ be such that

1. $i^{\tilde{m}_t} \in I(\hat{C}) \setminus \{j^*\}$ for all $t \in \{1, \ldots, T\}$
2. $\tilde{m}_t < \tilde{m}_{t+1}$ for all $t \leq T-1$ and $\tilde{m}_T \leq m^*$
3. For all $m' \in \{2, \ldots, m^*\} \setminus \{\tilde{m}_1, \ldots, \tilde{m}_T\}$, $i^{m'} \notin I(\hat{C})$

Let $\tilde{l}_1, \ldots, \tilde{l}_T$ be such that $j^{\tilde{l}_t} = i^{\tilde{m}_t}$ for all $t \in \{1, \ldots, T\}$. Note that since either $o^{m^*} = o \in O(\hat{C})$ or $m^* < \overline{m}$, there exists $t^* \leq T$ such that

$$\{p^{\tilde{l}_{t^*}}, j^{\tilde{l}_{t^*}+1}, p^{\tilde{l}_{t^*}+1}, \ldots [p^L, j^1] \ldots, j^{\tilde{l}_{t^*+1}-1}, p^{\tilde{l}_{t^*+1}-1}\} \cap \{o^{m^*}, i^{m^*+1}, o^{m^*+1}, \ldots, i^M, o^M\} \neq \emptyset, \quad (1)$$

where $\tilde{l}_{t^*+1} = \tilde{l}_1$ if $t^* = T$. For the remainder of the proof of Claim 6, fix some t^* for which Equation (1) holds and let $\hat{l}$ be the first integer in $(\tilde{l}_{t^*}, \ldots [L, 1] \ldots, \tilde{l}_{t^*+1} - 1)$ such that either $p^{\hat{l}} \in \{o^{m^*}, \ldots, o^M\}$ or $j^{\hat{l}} \in \{i^{m^*+1}, \ldots, i^M\}$.

We distinguish two cases:

Case 1: $\hat{l} \neq \tilde{l}_{t^*}$ and $j^* \in \{j^{\tilde{l}_{t^*}+1}, \ldots [j^L, j^1] \ldots, j^{\hat{l}-1}\}$.

It proves useful to further distinguish three subcases.

Case 1.1: $\boldsymbol{j^* = \hat{j}^* = j^1}$. If $j^{\hat{l}} \in \{i^{m^*+1}, \ldots, i^M\}$, let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$ and consider the sequence

$$\tilde{C} \equiv (j^1, p^1, \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}}, \ldots, i^M, o^M).$$

Note that $\hat{l} > 1$ since $j^{\hat{l}} \in \{i^{m^*+1}, \ldots, i^M\}$ and $j^1 = i^1 = j^*$ in the case we consider here. By the definitions of $\tilde{l}_{t^*}$ and $\tilde{l}_{t^*+1}$ as well as the fact that $\hat{l} \leq \tilde{l}_{t^*+1} - 1$, we obtain $\{p^1, \ldots, p^{\hat{l}-1}\} \cap O(C) = \emptyset$. We thus obtain that $\tilde{C}$ is a cycle of μ. As in the proofs of previous claims, it follows that $\tilde{C}$ is individually rational given that C and $\hat{C}$ are individually rational. Finally, since $p^1 \in A_{j^*}$, $o^M \in \Omega_{j^*} \setminus A_{j^*}$, and since, by Lemma 8, $\tilde{C}$ does not affect any agent $j' < j^*$, we obtain that $\tilde{C}$ is a CIC of μ for $j^* = j^1$ at A. Since $\hat{m} \geq m^* + 1$, we have that $o \notin O(\tilde{C})$. Hence, $\tilde{C}$ is a CIC of μ for j^* at $\hat{A}$, which is a contradiction to $\mu \in \mathcal{M}^{i-1}(\hat{A}) \subseteq \mathcal{M}^{j^*}(\hat{A})$. If $p^{\hat{l}} \in O(C)$ and $j^{\hat{l}} \notin I(C)$, let $\hat{m}$ be such that $o^{\hat{m}} = p^{\hat{l}}$ and consider

$$(j^1, p^1, \ldots, j^{\hat{l}}, p^{\hat{l}}, i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots, i^M, o^M)$$

to obtain a contradiction in a similar manner as before.[18]

The remaining two subcases deal with the possibility that $j^* \neq \hat{j}^*$. Let $\tilde{l} \geq 2$ be such that $j^{\tilde{l}} = j^*$ and note that Claim 3 implies $j^{\tilde{l}} < \hat{j}^*$.

Case 1.2: $\boldsymbol{p^{\hat{l}} \in A_{j^*}}$. If $j^{\hat{l}} \in \{i^{m^*+1}, \ldots, i^M\}$, then we must have $\hat{l} \neq \tilde{l}$. An analog of our arguments in the previous subcase implies that

$$(j^{\tilde{l}}, p^{\tilde{l}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$, which contradicts $\mu \in \mathcal{M}^{i-1}(\hat{A}) \subseteq \mathcal{M}^{j^*}(\hat{A})$. If $p^{\hat{l}} \in \{o^{m^*}, \ldots, o^M\}$ and $j^{\hat{l}} \notin I(C)$, let $\hat{m}$ be such that $o^{\hat{m}} = p^{\hat{l}}$ and consider

$$\tilde{C} \equiv (j^{\tilde{l}}, p^{\tilde{l}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}}, p^{\hat{l}}, i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots, i^M, o^M).\text{[19]}$$

Since i does not point to o in $\tilde{C}$, an analog of our earlier arguments implies that $\tilde{C}$ is a CIC of μ at $\hat{A}$, which is again impossible given our choice of μ.

Case 1.3: $\boldsymbol{p^{\hat{l}} \in \Omega_{j^*} \setminus A_{j^*}}$. Since $j^* < \hat{j}^*$ and since $\hat{C}$ is a CIC of $\mu + C$ for $\hat{j}^*$, we must have $p^{\tilde{l}-1} \in \Omega_{j^*} \setminus A_{j^*}$ as well. Consider the sequence

$$\tilde{C} \equiv (i^1, o^1, \ldots, i^{\tilde{m}_{t^*}-1}, o^{\tilde{m}_{t^*}-1}, j^{\tilde{l}_{t^*}}, p^{\tilde{l}_{t^*}}, \ldots [p^L, j^1] \ldots j^{\tilde{l}-1}, p^{\tilde{l}-1}).$$

Note that $\tilde{m}_{t^*} \geq 2$ and $\tilde{l}_{t^*} \neq \tilde{l}$ since $i^{\tilde{m}_{t^*}} \in I(C) \setminus \{j^*\}$ and $j^* = j^{\tilde{l}}$. We now show that $\{p^{\tilde{l}_{t^*}}, \ldots [p^L, p^1] \ldots, p^{\tilde{l}-1}\} \cap O(C) = \emptyset$. If $p^l \in O(C)$ for some $l \in \{\tilde{l}_{t^*}, \ldots [L, 1] \ldots, \tilde{l}-2\}$, then, since $\hat{C}$ is a CIC of $\mu + C$, we have $j^{l+1} \in I(C)$. Since $l \in \{l_{t^*}+1, \ldots [L, 1] \ldots, l_{t^*+1}-1\}$, the construction of the sequence $\{\tilde{l}_t\}_{t=1}^T$ implies that $j^{l+1} \in \{i^{m^*+1}, \ldots, i^M\}$. However, since $\tilde{l} \in \{\tilde{l}_{t^*}+1, \ldots [L, 1] \ldots, \hat{l}-1\}$ and $\{p^{\tilde{l}_{t^*}}, j^{\tilde{l}_{t^*}+1}, p^{\tilde{l}_{t^*}+1}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}\} \cap \{o^{m^*}, i^{m^*+1}, o^{m^*+1}, \ldots, i^M, o^M\} = \emptyset$, $j^{l+1} \in \{i^{m^*+1}, \ldots, i^M\}$ is impossible. If $p^{\tilde{l}-1} \in O(C)$, then, since $\hat{C}$ is a CIC of $\mu + C$, we must have $p^{\tilde{l}-1} = o^1$. However, since $p^{\tilde{l}-1} \in \Omega_{j^*} \setminus A_{j^*}$ and $o^1 \in A_{j^*}$ we have $p^{\tilde{l}-1} \neq o^1$. Given that we have now established $\{p^{\tilde{l}_{t^*}+1}, \ldots [p^L, p^1] \ldots, p^{\tilde{l}-1}\} \cap O(C) = \emptyset$, similar arguments as before show that $\tilde{C}$ satisfies all conditions of a CIC of μ for j^* at A. Furthermore, since $\tilde{m}_{t^*} - 1 < m^*$, we have that $o \notin O(\tilde{C})$ and $\tilde{C}$ is also a CIC of μ for j^* at $\hat{A}$, which is a contradiction.

Case 2: $\hat{l} = \tilde{l}_{t^*}$ or $\hat{l} \neq \tilde{l}_{t^*}$ and $j^* \notin \{j^{\tilde{l}_{t^*}+1}, \ldots [j^L, j^1] \ldots, j^{\hat{l}-1}\}$.

[18] The only difference to before is that the sequence may now contain o. However, given that $\hat{m} + 1 > m^*$, it cannot be the case that i points to o in the proposed cycle.

[19] Note that $\{i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots, i^M, o^M\} = \emptyset$ if $\hat{m} = M$.

Let $\tilde{m} \equiv \tilde{m}_{t^*}$, $\tilde{l} \equiv \tilde{l}_{t^*}$, and $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$ if $j^{\hat{l}} \in I(C)$ and $o^{\hat{m}} = p^{\hat{l}}$ if $j^{\hat{l}} \notin I(C)$. Note that the properties that define $\tilde{m}, \tilde{l}, \hat{l}, \hat{m}$ imply that if $p^{\hat{l}} \notin \{o^{m^*}, \ldots, o^M\}$, then $\hat{l} \neq \tilde{l}$ since $\tilde{m} \leq m^*$.[20]

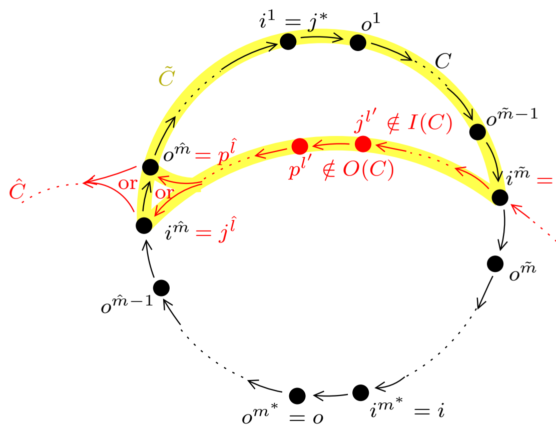


Figure 4: Construction of a CIC of μ for j^* at A.

If $p^{\hat{l}} \in \{o^{m^*}, \ldots, o^M\}$ and $j^{\hat{l}} \notin I(C)$, then we claim that

$$\tilde{C} \equiv (i^1, o^1, \ldots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}}, p^{\hat{l}}, i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots i^M, o^M)$$

is a CIC of μ for j^* at A.[21] The arguments to show that $\tilde{C}$ is a cycle of μ are analogous to the corresponding arguments in the proof of Lemma 8. Next, recall that, by Claim 3, $j^* \leq \hat{j}^*$. Hence, Lemma 8 implies that $\tilde{C}$ does not affect the welfare of any agent $k < j^* = \min\{j^*, \hat{j}^*\}$. Since $o^1 \in A_{j^*}$ as well as $o^M \in \Omega_{j^*} \setminus A_{j^*}$, and since $i^1 = j^*$ is the only instance of j^* in $\tilde{C}$, $\tilde{C}$ increases the welfare of j^*. Note that the definitions of $\tilde{m}, \tilde{l}, \hat{l}, \hat{m}$ imply that $i \in I(\tilde{C})$ only if $i = j^{\tilde{l}}$. However, since $\hat{C}$ is a CIC of $\mu + C$ and $o \in (\mu + C)(i) \setminus \mu(i)$, we must have $p^{\tilde{l}} \neq o$. Hence, $o \notin (\mu + \tilde{C})(i)$ and therefore $\tilde{C}$ is a CIC of μ for j^* at $\hat{A}$ as well. Again, this last finding contradicts the assumption that $\mu \in \mathcal{M}^{i-1}(\hat{A}) \subseteq \mathcal{M}^{j^*}(\hat{A})$.

If $j^{\hat{l}} \in I(C)$, then analogous arguments can be used to establish that

$$(i^1, o^1, \ldots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$ and we again obtain a contradiction.

We have thus shown that if $\underline{m} \leq m^*$, then $o \notin O(\hat{C})$ and $\overline{m} \leq m^*$.

[20] See Figure 4.

[21] Note that $\tilde{m} \geq 2$ and that $\{i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots i^M, o^M\} = \emptyset$ if $p^{\hat{l}} = o^M$. Note also that we must have $o^M \in O(\tilde{C})$.

Finally, we now show that $o \notin O(\hat{C})$. Assume to the contrary that there exists an l such that $p^l = o$. Since $\hat{C}$ is a CIC of $\mu + C$ at A and $o^{m^*} = o$, we have that $j^{l+1} = i^{m^*} = i$. Since $i \in I(\hat{C})$, $\underline{m} \leq m^*$ so that our previous arguments imply $o \notin O(\hat{C})$, thus contradicting our hypothesis. □

Proof of Claim 7. We proceed by contradiction. Without loss of generality, assume that m' is such that $i^{m'}$ is highest priority agent for whom one of the conditions in Claim 7 is violated. We distinguish six cases:

Case 1: $\boldsymbol{m' \in \{\underline{m}+1, \ldots, \overline{m}-1\}}$, $\boldsymbol{o^{m'-1} \in \Omega_{i^{m'}} \setminus A_{i^{m'}}}$, and $\boldsymbol{o^{m'} \in A_{i^{m'}}}$ We show that there exists a CIC of μ for $i^{m'}$ at $\hat{A}$. Since $i^{m'} < \hat{j}^* < i$ we obtain a contradiction to the assumption that $\mu \in \mathcal{M}^{i-1}(\hat{A}) \subseteq \mathcal{M}^{i^{m'}}(\hat{A})$.

Let $\tilde{m}_1$ be the smallest integer in $(m', \ldots, \overline{m})$ such that $i^{\tilde{m}_1} \in I(\hat{C})$. Note that such an integer must exist since $m' < \overline{m}$. Let $\tilde{l}_1$ be such that $j^{\tilde{l}_1} = i^{\tilde{m}_1}$. By definition of m' and $\overline{m}$, there exist $\tilde{l}_2, \ldots, \tilde{l}_T$ such that

1. $j^{\tilde{l}_t} \in \{i^{m'}, \ldots, i^{\overline{m}}\}$ for all $t \in \{1, \ldots, T\}$
2. $\{j^{\tilde{l}_t+1}, \ldots [j^L, j^1] \ldots, j^{\tilde{l}_{t+1}-1}\} \cap \{i^{m'}, \ldots, i^{\overline{m}}\} = \emptyset$ for all $t \in \{1, \ldots, T\}$, where $\tilde{l}_{T+1} \equiv \tilde{l}_1$
3. there exists no $l \in \{1, \ldots, L\} \setminus \{\tilde{l}_1, \ldots, \tilde{l}_T\}$ such that $j^l \in \{i^{m'}, \ldots, i^{\overline{m}}\}$

Note that since $m' > \underline{m}$ and $i^{\underline{m}} \in I(\hat{C})$, there exists a $t^* \in \{1, \ldots, T\}$ such that

$$\{j^{\tilde{l}_{t^*}+1}, \ldots [j^L, j^1] \ldots j^{\tilde{l}_{t^*+1}-1}\} \cap \{i^{\underline{m}}, \ldots, i^{m'-1}\} \neq \emptyset.\text{[22]} \tag{2}$$

We assume first that there is only one t^* for which (2) holds, that $t^* = 1$, and that $\tilde{m}_1$ is such that $\tilde{m}_1 = m'$. Here, we distinguish two subcases:

- **Case 1.1**: $p^{\tilde{l}_1} \in \Omega_{j^{\tilde{l}_1}} \setminus A_{j^{\tilde{l}_1}}$
  Since $j^{\tilde{l}_1} = i^{m'}$ and $i^{m'} < \hat{j}^*$, we must have $p^{\tilde{l}_1-1} \in \Omega_{j^{\tilde{l}_1}} \setminus A_{j^{\tilde{l}_1}}$ given that $\hat{C}$ is a CIC of $\mu + C$ for $\hat{j}^*$ at A. By our definition of the sequence $\{\tilde{l}_t\}_{t=1}^T$ and our assumption about t^*, we have $\{j^{\tilde{l}_T}, \ldots [j^L, j^1] \ldots j^{\tilde{l}_1-1}\} \cap I(C) = \{j^{\tilde{l}_T}\}$.[23] Furthermore, we also have that $\{p^{\tilde{l}_T}, \ldots [p^L, p^1] \ldots, p^{\tilde{l}_1-1}\} \cap O(C) = \emptyset$: for $l' \in \{\tilde{l}_T, \ldots [L, 1] \ldots, \tilde{l}_1 - 2\}$, $p^{l'} \in O(C)$ would imply that $j^{l'+1} \in I(C)$, which we have already argued is

[22] See Figure 5.

[23] Note that since $m' < \overline{m}$ and $j^{\tilde{l}_1} = i^{m'}$ in the case we consider here, we must have $\tilde{l}_T \neq \tilde{l}_1$.

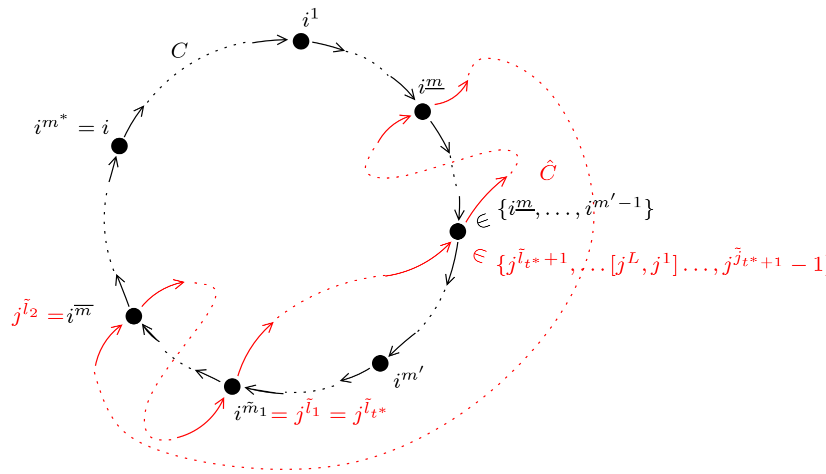


Figure 5: Definitions of the $\tilde{m}_1$, $\{\tilde{l}_t\}_{t=1}^T$, and t^*.

impossible; if $p^{\tilde{l}_1-1} \in O(C)$, then since $\hat{C}$ is a CIC of $\mu + C$ for $\hat{j}^*$ at A, we would have that $p^{\tilde{l}_1-1} = o^{m'}$, thus contradicting our assumption that $o^{m'} \in A_{i^{m'}}$ and $p^{\tilde{l}_1-1} \in \Omega_{i^{m'}} \setminus A_{i^{m'}}$.

Now let $\hat{m}$ be such that $i^{\hat{m}} = j^{\tilde{l}_T}$ and note that $\hat{m} \in \{m'+1, \dots, \overline{m}\}$ since $\tilde{l}_T \neq \tilde{l}_1$, $j^{\tilde{l}_1} = i^{m'}$, and all agents in $I(\hat{C})$ are distinct. Consider the following sequence of agents and objects

$$\tilde{C} \equiv (i^{m'}, o^{m'}, \dots, i^{\hat{m}-1}, o^{\hat{m}-1}, j^{\tilde{l}_T}, p^{\tilde{l}_T}, \dots [p^L, j^1] \dots, j^{\tilde{l}_1-1}, p^{\tilde{l}_1-1}).$$

We claim that $\tilde{C}$ is an CIC of μ for $i^{m'}$ at $\hat{A}$. By our arguments above, all agents and objects in $\tilde{C}$ are distinct. Since $\hat{C}$ is a CIC of $\mu + C$, $\{j^{\tilde{l}_T}, \dots [j^L, j^1] \dots, j^{\tilde{l}_1-1}\} \cap I(C) = \emptyset$ and $\{p^{\tilde{l}_T}, \dots [p^L, p^1] \dots, p^{\tilde{l}_1-1}\} \cap O(C) = \emptyset$, $\hat{C}$ is a cycle of μ. Since $i^{m'}$ is the highest priority agent for whom Claim 7 is violated, $\tilde{C}$ does not affect any agents with higher priority than $i^{m'}$.[24] Given that $o^{m'} \in A_{i^{m'}}$, $p^{\tilde{l}_1-1} \in \Omega_{i^{m'}} \setminus A_{i^{m'}}$, and $i^{m'}$ only appears once in $\tilde{C}$, the remaining requirements for $\tilde{C}$ to be a CIC of μ for $i^{m'}$ at $\hat{A}$ are also satisfied. Finally, since $\underline{m} < m'$ and $\hat{m} \leq \overline{m}$, Claim 6 implies that $o \notin \{o^{m'}, \dots, o^{\hat{m}-1}\}$. Since $\{p^{\tilde{l}_T}, \dots [p^L, p^1] \dots, p^{\tilde{l}_1-1}\} \cap O(C) = \emptyset$, we thus obtain that $o \notin O(\tilde{C})$. Hence, $\tilde{C}$ is an individually rational cycle of μ at $\hat{A}$. Combined with the previous findings, we obtain that $\tilde{C}$ is a CIC of μ for $i^{m'}$ at $\hat{A}$. Since $i^{m'} < \hat{j}^* < i$, we obtain $\mu \notin \mathcal{M}^{i-1}(\hat{A}) \subseteq \mathcal{M}^{i^{m'}}(\hat{A})$, which is a contradiction.

- **Case 1.2**: $p^{\tilde{l}_1} \in A_{j^{\tilde{l}_1}}$

[24] To see this, note that our assumptions about m' and $\hat{m}$ imply $\{i^{m'}, \dots, i^{\hat{m}}\} \subseteq \{i^{\underline{m}}, \dots, i^{\overline{m}}\}$. If there was an $m'' \in \{m', \dots, \overline{m}\}$ such that $\tilde{C}$ affects the welfare of $i^{m''} < i^{m'}$, either the first (if $m'' < \overline{m}$) or the third (if $m'' = \overline{m}$) condition of Claim 7 would be violated for $i^{m''}$, a contradiction to the definition of m'. Analogous arguments apply to the other places in the proof of Claim 7 in which we rely on the definition of m'.

Since $j^{\tilde{l}_1} = i^{\tilde{m}_1} = i^{m'}$ and $i^{m'} < \hat{j}^*$, we must have $p^{\tilde{l}_1 - 1} \in A_{j^{\tilde{l}_1}}$ given that $\hat{C}$ is a CIC of $\mu + C$ for $\hat{j}^*$ at A. Let $\hat{l}$ be the first integer in the sequence $(\tilde{l}_1 + 1, \ldots [L, 1] \ldots \tilde{l}_2 - 1)$ such that $j^{\hat{l}} \in \{i^{\underline{m}}, \ldots, i^{m'-1}\}$ and let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. Since we have assumed that $t^* = 1$, the definition of t^* implies that such $\hat{l}$ exists. By our definition of the sequence $\{\tilde{l}_t\}_{t=1}^{T}$ and our definition of $\hat{l}$, we have that $\{j^{\tilde{l}_1+1}, \ldots [j^L, j^1] \ldots j^{\hat{l}-1}\} \cap I(C) = \emptyset$. Next, note that $\{p^{\tilde{l}_1}, \ldots [p^L, p^1] \ldots, p^{\hat{l}-2}\} \cap O(C) = \emptyset$: if there is $\ddot{l}$ such that $p^{\ddot{l}} \in \{p^{\tilde{l}_1}, \ldots [p^L, p^1] \ldots, p^{\hat{l}-2}\} \cap O(C)$, then $j^{\ddot{l}+1} \in I(C)$ contradicting the earlier assertion that $\{j^{\tilde{l}_1+1}, \ldots [j^L, j^1] \ldots, j^{\hat{l}-1}\} \cap I(C) = \emptyset$ since $\ddot{l} \in \{\tilde{l}_1, \ldots [L, 1] \ldots, \hat{l} - 2\}$.

Now assume first that $p^{\hat{l}-1} \in O(C)$. Since $j^{\hat{l}} = i^{\hat{m}}$ and $\hat{C}$ is a CIC of $\mu + C$, we must have $p^{\hat{l}-1} = o^{\hat{m}}$. An analog of the arguments in Case 1.1 shows that

$$(j^{\tilde{l}_1}, p^{\tilde{l}_1}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots, i^{m'-1}, o^{m'-1})$$

is a CIC of μ for $i^{m'}$ at $\hat{A}$.

Similarly, if $p^{\hat{l}-1} \notin O(C)$, then

$$(j^{\tilde{l}_1}, p^{\tilde{l}_1}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}}, \ldots, i^{m'-1}, o^{m'-1})$$

is a CIC of μ for $i^{m'}$ at $\hat{A}$.

For the remainder of the proof of Case 1, we assume that there is a t^* for which (2) holds and $j^{\tilde{l}_{t^*}} \neq i^{m'}$. Let $\hat{l}$ be the first integer in $(\tilde{l}_{t^*}, \ldots [L, 1] \ldots, \tilde{l}_{t^*+1} - 1)$ such that $j^{\hat{l}} \in \{i^{\underline{m}}, \ldots, i^{m'-1}\}$ and let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. Assume first that $p^{\hat{l}-1} \in O(C)$. Since $\hat{C}$ is a CIC of $\mu + C$, we must have $p^{\hat{l}-1} = o^{\hat{m}}$ and an analog of our previous arguments shows that

$$(i^{m'}, o^{m'}, \ldots, i^{\tilde{m}_{t^*}-1}, o^{\tilde{m}_{t^*}-1}, j^{\tilde{l}_{t^*}}, p^{\tilde{l}_{t^*}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots, i^{m'-1}, o^{m'-1})$$

is a CIC of μ for $i^{m'}$ at $\hat{A}$. If $p^{\hat{l}-1} \notin O(C)$, then

$$(i^{m'}, o^{m'}, \ldots, i^{\tilde{m}_{t^*}-1}, o^{\tilde{m}_{t^*}-1}, j^{\tilde{l}_{t^*}}, p^{\tilde{l}_{t^*}}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}}, \ldots, i^{m'-1}, o^{m'-1})$$

is a CIC of μ for $i^{m'}$ at $\hat{A}$.

Case 2: $\boldsymbol{m' \in \{\underline{m} + 1, \ldots, \overline{m} - 1\}}$**,** $\boldsymbol{o^{m'-1} \in A_{i^{m'}}}$**, and** $\boldsymbol{o^{m'} \in \Omega_{i^{m'}} \setminus A_{i^{m'}}}$ We construct a CIC of $\mu + C$ for $i^{m'}$ at A and thus, given that $i^{m'} < \hat{j}^*$, contradict the assumption that $\hat{j}^*$ is the highest priority agent for whom there exists a CIC of $\mu + C$ at A.

Note first that since $m' > \underline{m}$ there exists a largest integer $\hat{m} \in \{\underline{m}, \ldots, m'-1\}$ such that $i^{\hat{m}} \in I(\hat{C})$. Let $\hat{l}$ be such that $j^{\hat{l}} = i^{\hat{m}}$ and note that, since C is a CIC and $\hat{m} \neq m$, $i^{\hat{m}} \neq i^{m'}$. Next, let $\tilde{l}$ be the first integer in the sequence $(\hat{l}+1, \ldots [L,1] \ldots, \hat{l}-1)$ such that $j^{\tilde{l}} \in \{i^{m'+1}, \ldots, i^{\overline{m}}\}$. Note that such an integer exists since $m' < \overline{m}$ and $j^{\hat{l}} \in \{i^{\underline{m}}, \ldots, i^{m'-1}\}$. Let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$.

Assume first that $i^{m'} \notin \{j^{\hat{l}}, \ldots [j^L, j^1] \ldots, j^{\tilde{l}}\}$ and consider the following sequence

$$\tilde{C} \equiv (i^{m'}, o^{m'-1}, \ldots, i^{\hat{m}+1}, o^{\hat{m}}, j^{\hat{l}}, p^{\hat{l}}, \ldots [p^L, j^1] \ldots, j^{\tilde{l}-1}, p^{\tilde{l}-1}, i^{\tilde{m}}, o^{\tilde{m}-1}, \ldots, i^{m'+1}, o^{m'}).$$

We claim that $\tilde{C}$ is a CIC of $\mu + C$ for $i^{m'}$ at A. Our definitions of $\hat{m}, \hat{l}, \tilde{l}, \tilde{m}$ together with the assumption that $i^{m'} \notin \{j^{\hat{l}}, \ldots [j^L, j^1] \ldots, j^{\tilde{l}}\}$ imply that all agents in $\tilde{C}$ are distinct: the definition of $\hat{m}$ implies that $\{i^{\hat{m}+1}, \ldots, i^{m'}\} \cap I(\hat{C}) \subseteq \{i^{m'}\}$; the definitions of $\tilde{m}$, $\tilde{l}$, and $\hat{l}$ imply that $\{i^{m'+1}, \ldots, i^{\tilde{m}}\} \cap \{j^{\hat{l}}, \ldots [j^L, j^1] \ldots, j^{\tilde{l}-1}\} = \emptyset$.

Next, we show that all objects in $\tilde{C}$ are also distinct. Consider first some $m \in \{\hat{m}+1, \ldots, m'-1\}$. If $o^m \in \{p^{\hat{l}}, \ldots [p^L, p^1] \ldots, p^{\tilde{l}}\}$, then, since $\hat{C}$ is a CIC of $\mu + C$, we would have $i^m \in I(\hat{C})$, thus contradicting the definition of $\hat{m}$ as the largest integer in $\{\underline{m}, \ldots, m'-1\}$ such that $i^{\hat{m}} \in I(\hat{C})$. Next, note that since $o^{\hat{m}} \in (\mu + C)(i^{\hat{m}})$ and $p^{\hat{l}} \notin (\mu + C)(i^{\hat{m}})$, we must have $o^{\hat{m}} \neq p^{\hat{l}}$. Hence, given that $\hat{C}$ is a CIC and $i^{\hat{m}} = j^{\hat{l}} \notin \{j^{\hat{l}}+1, \ldots [j^L, j^1] \ldots, j^{\tilde{l}}\}$, $o^{\hat{m}} \notin \{p^{\hat{l}}, \ldots [p^L, p^1] \ldots, p^{\tilde{l}-1}\}$. Finally, consider some $m \in \{m', \ldots, \tilde{m}-1\}$. If $o^m \in \{p^{\hat{l}}, \ldots [p^L, p^1] \ldots, p^{\tilde{l}-1}\}$, we would have that $i^m \in \{j^{\hat{l}+1}, \ldots [j^L, j^1] \ldots, j^{\tilde{l}-1}\}$, thus contradicting the definition of $\tilde{l}$. Next, note that, for all m, $o^m \in (\mu + C)(i^m)$ since C is a CIC of μ and that, for all l, $p^l \in (\mu + C)(j^{l+1})$. Hence, $\tilde{C}$ is a cycle of $\mu + C$. Since $i^{m'}$ is the highest priority agent for whom Claim 7 is violated, does not affect any agent with higher priority than $i^{m'}$. Given that $o^{m'-1} \in A_{i^{m'}}$, $o^{m'} \in \Omega_{i^{m'}} \setminus A_{i^{m'}}$, and $i^{m'}$ only appears once in $\tilde{C}$, $\tilde{C}$ is a CIC of $\mu + C$ for $i^{m'}$ at $\hat{A}$. This contradicts our choice of $\hat{C}$ and $\hat{j}^*$.

Next, assume that there is an integer $l' \in \{\hat{l}, \ldots [L, 1] \ldots, \tilde{l}\}$ such that $j^{l'} = i^{m'}$. We distinguish two subcases.

- **Case 2.1**: $p^{l'-1} \in \Omega_{j^{l'}} \setminus A_{j^{l'}}$

  An analog of our earlier arguments shows that in this case

  $$(i^{m'}, o^{m'-1}, \ldots, i^{\hat{m}+1}, o^{\hat{m}}, j^{\hat{l}}, p^{\hat{l}}, \ldots [p^L, j^1] \ldots, j^{l'-1}, p^{l'-1})$$

  is a CIC of $\mu + C$ for $i^{m'}$ at A.

- **Case 2.2**: $p^{l'-1} \in A_{j^{l'}}$

Since $i^{m'} < \hat{j}^*$ and since $\hat{C}$ is a CIC of $\mu + C$ for $\hat{j}^*$ at A, we have $p^{l'} \in A_{j^{l'}}$. An analog of our earlier arguments shows that in this case

$$(j^{l'}, p^{l'}, \dots [p^L, j^1] \dots, j^{\tilde{l}-1}, p^{\tilde{l}-1}, i^{\tilde{m}}, o^{\tilde{m}-1}, \dots, i^{m'+1}, o^{m'})$$

is a CIC of $\mu + C$ for $j^{l'} = i^{m'}$ at A.

Case 3: $\boldsymbol{m' \in \{\underline{m}, \dots, \overline{m} - 1\}}$, $\boldsymbol{i^{m'} = j^{l'}}$ for some $\boldsymbol{l'}$, $\boldsymbol{o^{m'} \in A_{i^{m'}}}$, and $\boldsymbol{\{p^{l'-1}, p^{l'}\} \subseteq \Omega_{i^{m'}} \setminus A_{i^{m'}}}$.

Let $\hat{l}$ be the last integer in $(l' + 1, \dots [L, 1] \dots, l' - 1)$ such that either $j^{\hat{l}} \in I(C)$ or $p^{\hat{l}} \in O(C)$. Note that we must have $p^{l'-1} \notin O(C)$: Otherwise, since $\hat{C}$ is a CIC of $\mu + C$, we would have $p^{l'-1} = o^{m'}$, contradicting the premise of this case. We now show that $p^{\hat{l}} \notin O(C)$: Otherwise, there is an m such that $o^m = p^{\hat{l}}$. Since $\hat{C}$ is a CIC of $\mu + C$, $i^m = j^{\hat{l}+1}$. Since $p^{l'-1} \notin O(C)$, $\hat{l} \neq l' - 1$. Thus, $j^{\hat{l}+1} \in I(C)$ and we obtain a contradiction to the definition of $\hat{l}$. So $p^{\hat{l}} \notin O(C)$ and $j^{\hat{l}} \in I(C)$.

For the remainder of the proof, let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. We distinguish three subcases:

Case 3.1: $\boldsymbol{\hat{m} > m'}$ An analog of our arguments in Case 1.1 shows that the following sequence[25] is a CIC of μ for $i^{m'}$ at $\hat{A}$:

$$(i^{m'}, o^{m'}, \dots, i^{\hat{m}-1}, o^{\hat{m}-1}, j^{\hat{l}}, p^{\hat{l}}, \dots [p^L, j^1] \dots j^{l'-1}, p^{l'-1}).$$

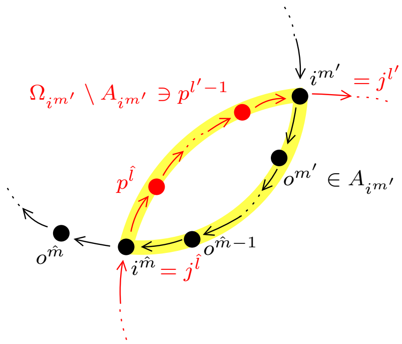
Case 3.2: $\boldsymbol{\hat{m} = m'}$ We obtain a contradiction since $\hat{C}$ is a CIC and hence $i^{m'} = j^{l'} \notin \{j^{l'+1}, \dots [j^L, j^1] \dots j^{l'-1}\}$.

Case 3.3: $\boldsymbol{\hat{m} < m'}$ Since $i^{\hat{m}} \in I(\hat{C})$ and $\hat{m} < m'$, we have $m' \neq \underline{m}$. By Case 1, we must then have that $o^{m'-1} \in A_{i^{m'}}$. Thus, an analog of our earlier arguments in Case 2 shows that the following sequence[26] is a CIC of $\mu + C$ for $i^{m'}$ at A:

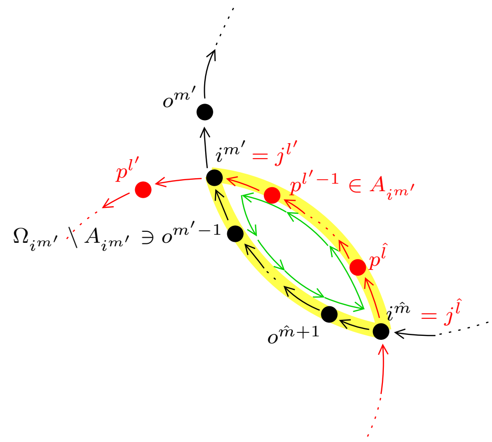
$$(i^{m'}, o^{m'-1}, \dots, i^{\hat{m}+2}, o^{\hat{m}+1}, j^{\hat{l}}, p^{\hat{l}}, \dots [p^L, j^1] \dots, j^{l'-1}, p^{l'-1}).$$

Case 4: $\boldsymbol{m' \in \{\underline{m}, \dots, \overline{m} - 1\}}$, $\boldsymbol{i^{m'} = j^{l'}}$ for some $\boldsymbol{l'}$, $\boldsymbol{o^{m'} \in \Omega_{i^{m'}} \setminus A_{i^{m'}}}$, and $\boldsymbol{\{p^{l'-1}, p^{l'}\} \subseteq A_{i^{m'}}}$

[25] See Figure 6a.
[26] See Figure 6b.



(a) As described in Case 3.1 in the proof of Claim 7, the highlighted path forms a CIC of μ for $i^{m'}$ at $\hat{A}$, the agent with highest priority for whom the statement of the claim is violated.



(b) As described in Case 3.3 in the proof of Claim 7, the highlighted path forms a CIC of $\mu + C$ for $i^{m'}$ at A, the agent with highest priority for whom the statement of the claim is violated. Notice that since this is a CIC of $\mu + C$, we use a section of C in reverse—that is, we undo some of what C does.

Figure 6: Illustrations of the cycles that lead to contradictions in the subcases of Case 3 in the proof of Claim 7.

Let $\hat{l}$ be the first integer in $(l'+1, \dots [L,1] \dots, l'-1)$ such that $j^{\hat{l}} \in I(C)$. Note that such an $\hat{l}$ exists since $\underline{m} \neq \overline{m}$. Note also that $\{p^{l'}, \dots [p^L, p^1] \dots, p^{\hat{l}-2}\} \cap O(C) = \emptyset$ since if, for some $l \in \{l', \dots [L,1] \dots, \hat{l}-2\}$, $p^l \in O(C)$, then $j^{l+1} \in I(C)$. Since $l+1 \in \{l'+1, \dots [L,1] \dots, \hat{l}-1\}$, we contradict the definition of $\hat{l}$. Let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. Note that if $p^{\hat{l}-1} \in O(C)$, then since $\hat{C}$ is a CIC of $\mu + C$, we must have $p^{\hat{l}-1} = o^{\hat{m}}$.

We now distinguish three subcases parallel to those of Case 3:

Case 4.1: $\hat{m} > m'$ An analog of our arguments in Case 2.2 shows that the following sequence[27] is a CIC of $\mu + C$ for $i^{m'}$ at A:

$$(j^{l'}, p^{l'}, \dots [p^L, j^1] \dots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}-1}, \dots, i^{m'+1}, o^{m'}).$$

Case 4.2: $\hat{m} = m'$ We obtain a contradiction since $\hat{C}$ is a CIC and hence $i^{m'} = j^{l'} \notin \{j^{l'+1}, \dots [j^L, j^1] \dots j^{l'-1}\}$.

Case 4.3: $\hat{m} < m'$ As in Case 3.3, $m' \neq \underline{m}$ and by appealing to Case 2 we conclude that $o^{m'-1} \in \Omega_{i^{m'}} \setminus A_{i^{m'}}$. Note also that since $m' < \overline{m}$, we must have $o \notin$

[27] See Figure 7a.

$\{o^{\hat{m}}, \ldots, o^{m'-1}\}$ by Claim 6.

Assume first that $p^{\hat{l}-1} \in O(C)$. As explained above, we must have $p^{\hat{l}-1} = o^{\hat{m}}$. An analog of our arguments in Case 1.2 shows that the sequence[28]

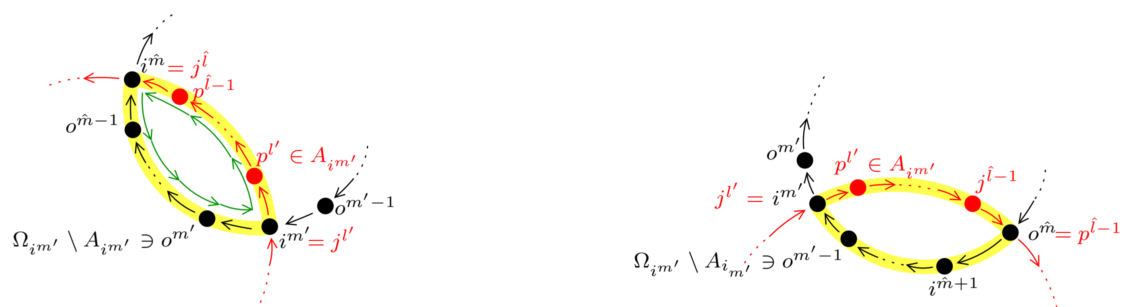
$$(j^{l'}, p^{l'}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots, i^{m'-1}, o^{m'-1})$$

is a CIC of μ for $i^{m'}$ at $\hat{A}$.

Next, consider the case where $p^{\hat{l}-1} \notin O(C)$. Again, an analog of our earlier arguments in Case 1.2 shows that the sequence

$$(j^{l'}, p^{l'}, \ldots [p^L, j^1] \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}}, \ldots, i^{m'-1}, o^{m'-1})$$

is a CIC of μ for $i^{m'}$ at $\hat{A}$.



(a) As described in Case 4.1 in the proof of Claim 7, the highlighted path forms a CIC of $\mu + C$ at A for $i^{m'}$, the agent with highest priority for whom the statement of the claim is violated. Notice that since this is a CIC of $\mu + C$, we use a section of C in reverse—that is, we undo some of what C does.

(b) As described in Case 4.3 in the proof of Claim 7, the highlighted path forms a CIC of μ for $i^{m'}$ at $\hat{A}$, the agent with highest priority for whom the statement of the claim is violated.

Figure 7: Illustrations of the cycles that lead to contradictions in the subcases of Case 4 in the proof of Claim 7.

Case 5: $m' = \overline{m}$, $i^{m'} = j^{l'}$ for some l', $o^{m'-1} \in A_{i^{m'}}$, and $\{p^{l'-1}, p^{l'}\} \subseteq \Omega_{i^{m'}} \setminus A_{i^{m'}}$.

Define $\hat{m}$ and $\hat{l}$ as in the proof of Case 3. Given that $m' = \overline{m}$, we must have $\hat{m} < m'$ and can hence apply the arguments in the proof of Case 3.3 (which only used the assumption that $m' < \overline{m}$ to infer that $o^{m'-1} \in A_{i^{m'}}$).

Case 6: $m' = \overline{m}$, $i^{m'} = j^{l'}$ for some l', $o^{m'-1} \in \Omega_{i^{m'}} \setminus A_{i^{m'}}$, and $\{p^{l'-1}, p^{l'}\} \subseteq A_{i^{m'}}$.

[28] See Figure 7b.

Define $\hat{m}$ and $\hat{l}$ as in the proof of Case 4. Given that $m' = \overline{m}$, we must have $\hat{m} < m'$. Furthermore, given that $i^{m'} < \hat{j}^* \leq i$, we must have $i^{m'} \neq \hat{j}^*$. Hence, Claim 6 implies that $i \notin \{i^{\hat{m}}, \dots, i^{m'}\}$ and $o \notin \{o^{\hat{m}}, \dots, o^{m'}\}$. We can thus apply the arguments in the proof of Case 4.3, which other than $o \notin \{o^{\hat{m}}, \dots, o^{m'}\}$, only used the assumption that $m' < \overline{m}$ to infer that $o^{m'-1} \in \Omega_{i^{m'}} \setminus A_{i^{m'}}$.

□

Proof of Claim 8. Assume to the contrary that l^* and $\check{m}$ with the properties listed in Claim 8 exist. Without loss of generality, assume that there is no m' such that $i^{m'} < i^{\check{m}}$ that also has the properties listed in Claim 8. We distinguish two cases:

Case 1: $\boldsymbol{o^{\check{m}-1} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}}$ and $\boldsymbol{o^{\check{m}} \in A_{i^{\check{m}}}}$. We argue first that $m^* \geq \overline{m}$. Assume to the contrary and let $\tilde{l}$ be the first integer in $(l^*+1, \dots [L,1] \dots, l^*-1)$ such that $j^{\tilde{l}} \in I(C)$. Note that such an integer exists since $(I(C) \cap I(\hat{C})) \setminus \{j^*\} \neq \emptyset$. If $p^{\tilde{l}-1} \notin O(C)$, let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$ and consider

$$\tilde{C} \equiv (j^{l^*}, p^{l^*}, \dots [j^L, p^1] \dots, j^{\tilde{l}-1}, p^{\tilde{l}-1}, i^{\tilde{m}}, o^{\tilde{m}}, \dots, i^M, o^M).$$

Since $p^{l^*} \in A_{j^*}$, analogs of our earlier arguments can be used to show that $\tilde{C}$ is a CIC of μ for $j^* = j^{l^*}$ at A. Since we assume that $m^* \not\geq \overline{m}$, Claim 6 implies $m^* < \underline{m}$. But then, we obtain that $m^* < \tilde{m}$ and $\tilde{C}$ is a CIC of μ for j^* at $\hat{A}$, which contradicts our assumption that $\mu \in \mathcal{M}^{i-1}(\hat{A})$ given that $j^* < i$. If $p^{\tilde{l}-1} = o^{\tilde{m}-1}$ for some $\tilde{m}$, an analogous argument applies.

Now let $\hat{l}$ be the last integer in $(l^*+1, \dots [L,1] \dots, l^*-1)$ such that $j^{\hat{l}} \in I(C)$. If $j^{\hat{l}} = i^{\check{m}}$ and $p^{l^*-1} \neq o^1$, consider

$$\tilde{C}' \equiv (j^{\hat{l}}, p^{\hat{l}}, \dots [p^L, j^1] \dots, j^{l^*-1}, p^{l^*-1}, i^1, o^1, \dots, i^{\check{m}-1}, o^{\check{m}-1}).$$

Since $j^{\hat{l}} = i^{\check{m}}$, and $\check{m} \in \{2, \dots, \underline{m}\}$, we have $\check{m} = \underline{m}$. By definition of $\hat{l}$, all agents in $I(\tilde{C}')$ are distinct. Since $\check{m}$ is the smallest integer for which the properties listed in Claim 8 are satisfied, $\tilde{C}'$ does not affect the welfare of any agent $j' \in \{i^2, \dots, i^{\check{m}-1}\}$ such that $j' < i^{\check{m}}$. Since $\{p^{l^*-1}, o^1\} \subseteq A_{j^*}$, $\tilde{C}'$ also does not affect the welfare of j^*. Since $\check{m} = \underline{m}$ and since C improves the welfare of $i^{\check{m}}$, Claim 7 implies that $p^{\hat{l}} \in A_{i^{\check{m}}}$. Hence, $\tilde{C}'$ is a CIC of μ for $i^{\check{m}}$ at A. Since $\check{m} = \underline{m}$ and $m^* \geq \overline{m}$, we obtain that $o \notin O(\tilde{C}')$. Combining our last two observations, we obtain that $\tilde{C}'$ is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$, which contradicts $\mu \in \mathcal{M}^{i-1}(\hat{A})$ given that $i^{\check{m}} < \hat{j}^* < i$. If $j^{\hat{l}} = i^{\check{m}}$ and $p^{l^*-1} = o^1$, we obtain a similar contradiction using

$$(j^{\hat{l}}, p^{\hat{l}}, \dots [p^L, j^1] \dots, j^{l^*-1}, p^{l^*-1}, i^2, o^2, \dots, i^{\check{m}-1}, o^{\check{m}-1}).$$

If $j^{\hat{l}} \neq i^{\check{m}}$, let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. Since $\check{m} \in \{2, \dots, \underline{m}\}$, $\hat{m} > \check{m}$. If $p^{l^*-1} \neq o^1$, analogs of our earlier arguments show that

$$(i^1, o^1, \dots, i^{\hat{m}-1}, o^{\hat{m}-1}, j^{\hat{l}}, p^{\hat{l}}, \dots [p^L, j^1] \dots j^{l^*-1}, p^{l^*-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$. Again, if $p^{l^*-1} = o^1$, we obtain a similar contradiction using

$$(i^2, o^2, \dots, i^{\hat{m}-1}, o^{\hat{m}-1}, j^{\hat{l}}, p^{\hat{l}}, \dots [p^L, j^1] \dots j^{l^*-1}, p^{l^*-1}).$$

Case 2: $o^{\check{m}-1} \in A_{i^{\check{m}}}$ and $o^{\check{m}} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$. Let $\hat{l}$ be the first integer in $(l^*+1, \dots [L, 1] \dots, l^*-1)$ such that $j^{\hat{l}} \in I(C)$.

If $j^{\hat{l}} = i^{\check{m}}$, the conditions of Claim 8 imply that $\check{m} = \underline{m}$. Since $o^{\underline{m}} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$, the second part of Claim 7 implies $p^{\hat{l}-1} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$. Consider

$$\tilde{C} \equiv (i^{\check{m}}, o^{\check{m}-1}, \dots, i^2, o^1, j^{l^*}, p^{l^*}, \dots [p^L, j^1] \dots, j^{\hat{l}-1}, p^{\hat{l}-1}).$$

By definition of $\hat{l}$ and our assumption about $\check{m}$, $\tilde{C}$ is a CIC of $\mu + C$ for $i^{\check{m}}$ at A. Since $i^{\check{m}} < \hat{j}^*$, we obtain a contradiction to our assumption that $\hat{j}^*$ is the highest priority agent for whom there exists a CIC of $\mu + C$ at A.

If $j^{\hat{l}} \neq i^{\check{m}}$, let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. Since $\check{m} \in \{2, \dots, \underline{m}\}$, $\check{m} < \hat{m}$. Again, analogs of our earlier arguments establish that

$$\tilde{C}' \equiv (i^{\hat{m}}, o^{\hat{m}-1}, \dots, i^2, o^1, j^{l^*}, p^{l^*}, \dots [p^L, j^1] \dots, j^{\hat{l}-1}, p^{\hat{l}-1})$$

is a CIC of $\mu + C$ for $i^{\check{m}}$ at A, which contradicts our assumption that $\hat{j}^*$ is the highest priority agent for whom there exists a CIC of $\mu + C$ at A

□

Proof of Claim 9. Assume to the contrary that l^* and $\check{m}$ with the properties listed in Claim 9 exist. Without loss of generality, assume that there is no m' such that $i^{m'} < i^{\check{m}}$ that also has the properties listed in Claim 9. We distinguish two cases:

Case 1: $o^{\check{m}-1} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$ and $o^{\check{m}} \in A_{i^{\check{m}}}$. We argue first that we must have $m^* \leq \underline{m}$. Assume the contrary and let $\tilde{l}$ be the last integer in $(l^* + 1, \dots [L, 1] \dots, l^* - 1)$ such that $j^{\tilde{l}} \in I(C)$. Note that such an integer exists since $(I(C) \cap I(\hat{C})) \setminus \{j^*\} \neq \emptyset$. If $p^{\tilde{l}-1} \notin O(C)$, let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$ and consider

$$\tilde{C} \equiv (i^1, o^1, \ldots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \ldots [j^L, p^1] \ldots, j^{l^*-1}, p^{l^*-1}).$$

Since $p^{l^*-1} \in \Omega_{j^*} \setminus A_{j^*}$, analogs of our earlier arguments can be used to show that $\tilde{C}$ is a CIC of μ for $j^* = i^1$ at A. Since we assume that $m^* \not\leq \underline{m}$, Claim 6 implies that $m^* \geq \overline{m}$. But then, we obtain that $m^* \geq \tilde{m}$ and $\tilde{C}$ is a CIC of μ for j^* at $\hat{A}$, which contradicts our assumption that $\mu \in \mathcal{M}^{i-1}(\hat{A})$ given that $j^* < i$. If $p^{\tilde{l}-1} = o^{\tilde{m}-1}$ for some $\tilde{m}$, an analogous argument applies.

Now let $\hat{l}$ be the first integer in $(l^* + 1, \ldots [L, 1] \ldots, l^* - 1)$ such that $j^{\hat{l}} \in I(C)$. If $j^{\hat{l}} = i^{\check{m}}$ and $p^{l^*} \neq o^M$, consider

$$\tilde{C}' \equiv (i^{\check{m}}, o^{\check{m}}, \ldots, i^M, o^M, j^{l^*}, p^{l^*}, \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}).$$

Since $j^{\hat{l}} = i^{\check{m}}$ and $\check{m} \in \{\overline{m}, \ldots, M\}$, we have $\check{m} = \overline{m}$. By definition of $\hat{l}$, all agents in $I(\tilde{C}')$ are distinct. Since $\check{m}$ is the smallest integer with the properties listed in Claim 8, $\tilde{C}'$ does not affect any agent $j' \in \{i^{\check{m}+1}, \ldots, i^M\}$ such that $j' < i^{\check{m}}$. Since $\{o^M, p^{l^*}\} \subseteq \Omega_{j^*} \setminus A_{j^*}$, $\tilde{C}'$ also does not affect j^*. Since $\check{m} = \overline{m}$ and since C increases the welfare of $i^{\check{m}} < \hat{j}$, Claim 7 implies that $p^{\hat{l}-1} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}$ and, given that $o^{\check{m}} \in A_{i^{\check{m}}}$, $p^{\hat{l}-1} \notin O(C)$. Hence, $\tilde{C}'$ is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$. Since $\check{m} = \overline{m}$ and $m^* \leq \underline{m}$ as well as $m^* < \overline{m}$, we obtain that $o \notin O(\tilde{C}')$. Combining our last two observations, we obtain that $\tilde{C}'$ is a CIC of μ for $i^{\check{m}}$ at A, which contradicts $\mu \in \mathcal{M}^{i-1}(\hat{A})$ given that $i^{\check{m}} < \hat{j}^* < i$. If $j^{\hat{l}} = i^{\check{m}}$ and $p^{l^*} = o^M$, we obtain a similar contradiction using

$$(i^{\check{m}}, o^{\check{m}}, \ldots, i^{M-1}, o^{M-1}, j^{l^*+1}, p^{l^*+1}, \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}).$$

If $j^{\hat{l}} \neq i^{\check{m}}$, let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. Since $\check{m} \in \{\overline{m}, \ldots, M\}$, we have $\hat{m} < \check{m}$. If $p^{\hat{l}-1} \notin O(C)$ and $p^{l^*} \neq o^M$, analogs of our earlier arguments show that

$$(i^{\hat{m}}, o^{\hat{m}}, \ldots, i^M, o^M, j^{l^*}, p^{l^*}, \ldots, j^{\hat{l}-1}, p^{\hat{l}-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$. If $p^{\hat{l}-1} \notin O(C)$ and $p^{l^*} = o^M$, we obtain a similar contradiction using

$$(i^{\hat{m}}, o^{\hat{m}}, \ldots, i^{M-1}, o^{M-1}, j^{l^*+1}, p^{l^*+1}, \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}).$$

If $p^{\hat{l}-1} \in O(C)$ and $p^{l^*} \neq o^M$, analogs of our earlier arguments show that

$$(i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots, i^M, o^M, j^{l^*}, p^{l^*}, \ldots, j^{\hat{l}-1}, p^{\hat{l}-1})$$

is a CIC of μ for $i^{\check{m}}$ at $\hat{A}$. Finally, if $p^{\hat{l}-1} \in O(C)$ and $p^{l^*} = o^M$, we obtain a similar contradiction using

$$(i^{\hat{m}+1}, o^{\hat{m}+1}, \ldots, i^{M-1}, o^{M-1}, j^{l^*+1}, p^{l^*+1}, \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}).$$

Case 2: $\boldsymbol{o^{\check{m}-1} \in A_{i^{\check{m}}}}$ and $\boldsymbol{o^{\check{m}} \in \Omega_{i^{\check{m}}} \setminus A_{i^{\check{m}}}}$. Let $\hat{l}$ be the last integer in $(l^*+1, \ldots [L, 1] \ldots, l^*-1)$ such that $j^{\hat{l}} \in I(C)$.

If $j^{\hat{l}} = i^{\check{m}}$, the conditions of Claim 9 imply that $\check{m} = \overline{m}$. Since $o^{\overline{m}-1} \in A_{i^{\check{m}}}$, the third part of Claim 7 implies $p^{\hat{l}} \in A_{i^{\check{m}}}$. Consider

$$\tilde{C} \equiv (j^{\hat{l}}, p^{\hat{l}}, \ldots, j^{l^*-1}, p^{l^*-1}, i^1, o^M, \ldots, i^{\overline{m}+1}, o^{\overline{m}}).$$

By definition of $\hat{l}$ and our assumption about $\check{m}$, $\tilde{C}$ is a CIC of $\mu + C$ for $i^{\check{m}}$ at A. Since $i^{\check{m}} < \hat{j}^*$, we obtain a contradiction to our assumption that $\hat{j}^*$ is the highest priority agent for whom there exists a CIC of $\mu + C$ at A.

If $j^{\hat{l}} \neq i^{\check{m}}$, let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. Since $\check{m} \in \{\overline{m}, \ldots, M\}$, $\hat{m} < \check{m}$. Again, analogs of our earlier arguments establish that

$$\tilde{C}' \equiv (j^{\hat{l}}, p^{\hat{l}}, \ldots, j^{l^*-1}, p^{l^*-1}, i^1, o^M, \ldots, i^{\hat{m}+1}, o^{\hat{m}})$$

is a CIC of $\mu + C$ for $i^{\check{m}}$ at A, which contradicts our assumption that $\hat{j}^*$ is the highest priority agent for whom there exists a CIC of $\mu + C$ at A

□

Proof of Claim 10.

If C increases the welfare of $\hat{j}^*$, there exists an m' such that $i^{m'} = \hat{j}^*$, $o^{m'-1} \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$, and $o^{m'} \in A_{\hat{j}^*}$.

We argue first that $p^{\bar{l}} \notin O(C)$ and $j^{\bar{l}} \in I(C) \setminus \{\hat{j}^*\}$: otherwise, Claim 5 implies $\bar{l} = L$; but since $p^L \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$ and $o^{m'} \in A_{\hat{j}^*}$, we must have $o^{m'} \neq p^L$ and hence $p^L \notin O(C)$.

For the remainder of the proof of Claim 10, it is helpful to recall that the definitions of $\bar{l}$, $m(\bar{l})$ and $p^{\bar{l}} \notin O(C)$ imply $\{p^{\bar{l}}, \ldots, p^L\} \cap O(C) = \emptyset$ as well as $i^{m(\bar{l})+1} = j^{\bar{l}}$.

Next, we establish that $\hat{j}^* \neq j^*$. Suppose the contrary. If $m^* \geq m(\bar{l})+1$, analogs of our arguments for the cases already considered show that

$$(i^1, o^1, \ldots, i^{m(\bar{l})}, o^{m(\bar{l})}, j^{\bar{l}}, p^{\bar{l}}, \ldots, j^L, p^L).$$

is a CIC of μ for j^* at $\hat{A}$, thus contradicting $\mu \in \mathcal{M}^{i-1}(\hat{A})$. If $m^* < m(\bar{l})+1$, then Claim 6 implies that $m^* < \underline{m}$.

If $p^{\underline{l}} \notin O(C)$ and $j^{\underline{l}+1} = i^{m(\underline{l})}$, we must have $i^{m(\underline{l})} \neq j^*$: otherwise, $\underline{l} = L$ and $I(C) \cap I(\hat{C}) = \{j^*\}$, which we have already ruled out above. Hence, $m(\underline{l}) \geq \underline{m}$ and, given that $m^* < \underline{m}$, we obtain that

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$, which is impossible.

If $p^{\underline{l}} \in O(C)$, we must have $o^{m(\underline{l})-1} = p^{\underline{l}}$. We argue first that $m(\underline{l}) \geq 3$: if $m(\underline{l}) = 2$, we obtain that $o^1 \in O(\hat{C})$, which is impossible given that $j^* = \hat{j}^*$, $p^L \in \Omega_{j^*} \setminus A_{j^*}$, and $o^1 \in A_{j^*}$; if $m(\underline{l}) = 1$, we obtain $p^{\underline{l}} = o^0 = o^M$ and

$$(j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}})$$

is a CIC of μ for j^* at $\hat{A}$, which contradicts $\mu \in \mathcal{M}^{i-1}(\hat{A})$; if $m(\underline{l}) \geq 3$, we have that $i^{m(\underline{l})-1} \in I(\hat{C}) \setminus \{j^*\}$ and hence $m(\underline{l}) \geq \underline{m}$. Hence, we can proceed analogously to the case of $p^{\underline{l}} \notin O(C)$.

Thus, we have established that $\hat{j}^* \neq j^*$, so $m' \geq 2$. Note that $o^{m'} \in A_{\hat{j}^*}$ and $p^L \in \Omega_{\hat{j}^*} \setminus A_{\hat{j}^*}$ still imply $p^L \notin O(C)$.

We distinguish two cases.

Case 1: $\{i^{m'+1}, \ldots, i^M\} \cap I(\hat{C}) \neq \emptyset$. In this case, $\overline{m} > m'$.

Let $\tilde{l}$ be the last integer in $(\bar{l}+1, \ldots, L)$ such that $j^{\tilde{l}} \in \{i^{m'+1}, \ldots, i^M\}$ and let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$. Then, $\underline{m} \leq m'$ since $\hat{j}^* = i^{m'}$ and $j^* \neq \hat{j}^*$. Furthermore, the definition of $\tilde{m}$ implies $\overline{m} \geq \tilde{m} > m'$. Given that $m^* \neq m'$, Claim 6 implies $m^* \notin \{m', \ldots, \tilde{m}-1\}$. We use this observation repeatedly in our proof.

If $\{j^{\tilde{l}+1}, \ldots, j^L\} \cap I(C) = \emptyset$, then Claim 7 and $\underline{m} \leq m' < \tilde{m} \leq \overline{m}$ are, by now, easily seen to imply that

$$(i^{m'}, o^{m'}, \ldots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \ldots, j^L, p^L).$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, thus contradicting $\mu \in \mathcal{M}^{i-1}(\hat{A})$.

For the remainder of the proof in case $\{i^{m'+1}, \ldots, i^M\} \cap I(\hat{C}) \neq \emptyset$, let $\tilde{l}'$ be the first integer in $\tilde{l}+1, \ldots, L)$ such that $j^{\tilde{l}'} \in I(C)$ and let $\tilde{m}'$ be such that $i^{\tilde{m}'} = j^{\tilde{l}'}$. We distinguish three subcases.

Case 1.1: $\boldsymbol{i^{\tilde{m}'} = j^*}$ and $\boldsymbol{p^{\tilde{l}'} \in A_{j^*}}$. Assume first that $p^{\tilde{l}'-1} \in O(C)$. Since all the agents in $I(C)$ are distinct, we must have $p^{\tilde{l}'-1} = o^1$. We claim that it has to be the case that $m^* < \underline{m}$: if $m^* \geq \tilde{m}$, then

$$(i^2, o^2, \ldots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \ldots, j^{\tilde{l}'-1}, p^{\tilde{l}'-1})$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$ given that Claims 7 and 8 jointly imply that there cannot be an $m \leq \tilde{m}-1 < \overline{m}$ such that $i^m < \hat{j}^*$ and such that C increases the welfare of i^m; since $m^* < \tilde{m} \leq \overline{m}$, Claim 6 implies $m^* < \underline{m}$. This contradicts $\mu \in \mathcal{M}^{i-1}(\hat{A})$.

Now let $\hat{l}$ be the first integer in $(\tilde{l}', \ldots, L)$ such that either $j^{\hat{l}} \in I(C) \setminus \{j^*, \hat{j}^*\}$ or $p^{\hat{l}} \in O(C) \cup \{p^L\}$. A $\hat{l}$ with the desired properties exists since we allow for the case where $\hat{l} = L$, $j^L \notin I(C) \setminus \{j^*, \hat{j}^*\}$, and $p^L \notin O(C)$.

Consider first the case where $j^{\hat{l}} \in I(C) \setminus \{j^*, \hat{j}^*\}$ and let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$. Note that $\hat{l} > \tilde{l}' > \tilde{l}$ and thus, by definition of $\tilde{l}$, $\{j^{\tilde{l}+1}, \ldots, j^L\} \cap \{i^{m'+1}, \ldots, i^M\} = \emptyset$. Hence, we must have $\hat{m} \in \{2, \ldots, m'-1\}$ and

$$(j^{\tilde{l}'}, p^{\tilde{l}'}, \ldots, j^{\hat{l}-1}, p^{\hat{l}-1}, i^{\hat{m}}, o^{\hat{m}}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$ given that $m^* < \underline{m}$, which is impossible.

Next, consider the case where $j^{\hat{l}} \notin I(C) \setminus \{j^*, \hat{j}^*\}$ and $p^{\hat{l}} \in O(C)$. Let $\hat{m}$ be such that $o^{\hat{m}-1} = p^{\hat{l}}$ and consider

$$\tilde{C} \equiv (j^{\tilde{l}'}, p^{\tilde{l}'}, \ldots, j^{\hat{l}}, p^{\hat{l}}, i^{\hat{m}}, o^{\hat{m}}, \ldots, i^M, o^M).$$

Note that $\hat{l} \geq \tilde{l}'$ implies $p^{\hat{l}} \neq o^1$. Hence, either $\hat{m} = 1$ or $\hat{m} \geq 3$ and, given that $i^{\hat{m}-1} = j^{\hat{l}+1}$, $\hat{m} > \underline{m}$. Again, $m^* < \underline{m}$ implies that $\tilde{C}$ is a CIC of μ for j^* at $\hat{A}$, which is impossible.

Finally, in case $j^{\hat{l}} \notin I(C) \setminus \{j^*, \hat{j}^*\}$ and $p^{\hat{l}} \notin O(C)$, we must have $\hat{l} = L$. As in the previous case, we find that

$$(j^{\tilde{l}'}, p^{\tilde{l}'}, \ldots, j^L, p^L, i^{m'}, o^{m'}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$, which is impossible.

Now suppose that $p^{\tilde{l}-1} \notin O(C)$. If $m^* \geq \tilde{m}$, then Claims 7 and 8 imply that

$$(i^1, o^1, \dots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \dots, j^{\tilde{l}'-1}, p^{\tilde{l}'-1})$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, which is impossible. Thus, $m^* < \tilde{m}$. We now reach contradictions in the by the same argument as when $p^{\tilde{l}'-1} \in O(C)$.

Case 1.2: $\boldsymbol{i^{\tilde{m}'} = j^*}$ and $\boldsymbol{p^{\tilde{l}'-1} \in \Omega_{j^*} \setminus A_{j^*}}$. Since $o^1 \in A_{j^*}$ we have $p^{\tilde{l}'-1} \notin O(C)$.

As in the previous subcase, let $\hat{l}$ be the first integer in $(\tilde{l}', \dots, L)$ such that either $j^{\hat{l}} \in I(C) \setminus \{j^*\}$ or $p^{\hat{l}} \in O(C) \cup \{p^L\}$.

If $j^{\hat{l}} \in I(C) \setminus \{j^*\}$, let $\hat{m}$ be such that $i^{\hat{m}} = j^{\hat{l}}$ and consider

$$\tilde{C} \equiv (i^{\hat{m}}, o^{\hat{m}}, \dots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \dots, j^{\hat{l}-1}, p^{\hat{l}-1}).$$

The definitions of $\tilde{l}$, $\tilde{l}'$, and $\hat{l}$ imply that $\{p^{\tilde{l}}, \dots, p^{\hat{l}-1}\} \cap O(C) \subsetneq \{p^{\tilde{l}'-1}\}$. Since $p^{\tilde{l}'-1} \notin O(C)$, analogs of our earlier arguments show that $\tilde{C}$ is a cycle of μ.

Next, we show that $\hat{m} < m' < \tilde{m}$. First, since we have defined $\tilde{l}$ such that $j^{\tilde{l}} \in \{i^{m'+1}, \dots, i^M\}$ and since $i^{\tilde{m}} = j^{\tilde{l}}$, we have that $m' < \tilde{m}$. Now we show that $m' > \hat{m}$. By definition of $\tilde{l}$ and $\tilde{l}'$, we know that $\hat{l} > \tilde{l}' > \tilde{l}$. By definition of $\tilde{l}$, $\{j^{\tilde{l}+1}, \dots, j^L\} \cap \{i^{m'+1}, \dots, i^M\} = \emptyset$. Thus, $m' \geq \hat{m}$. Since $\hat{l} > 1$, $i^{\hat{m}} = j^{\hat{l}} \neq j^1 = \hat{j}^* = i^{m'}$. Thus, $m' \neq \hat{m}$ so that $m' > \hat{m}$.

Given that $\underline{m} \leq \hat{m} \leq \tilde{m} - 1 < \overline{m}$, Claim 6 implies $m^* \notin \{\hat{m}, \dots, \tilde{m} - 1\}$.

Now we use the preceding observations to show that $\tilde{C}$ is a CIC of μ for $\hat{j}^*$ at $\hat{A}$. Since $\hat{m} < m' < \tilde{m}$ and since C increases the welfare of $\hat{j}^*$, $\tilde{C}$ increases the welfare of $\hat{j}^*$. Since $\tilde{l} < \tilde{l}' < \hat{l}$ and since j^* exchanges two undesirable objects on $\hat{C}$, $\tilde{C}$ does not affect the welfare of j^*. For all $m \in \{\underline{m}+1, \dots, \overline{m}-1\}$ such that $i^m < \hat{j}^*$, the first part of Claim 7 implies that C, and therefore also $\tilde{C}$, does not affect the welfare of i^m. Next, if $\hat{m} = \underline{m}$ and $i^{\underline{m}} < \hat{j}^*$, then the second part of Claim 7 implies that either $\{p^{\hat{l}-1}, o^{\hat{m}}\} \subseteq \Omega_{i^{\underline{m}}} \setminus A_{i^{\underline{m}}}$ or $\{p^{\hat{l}-1}, o^{\hat{m}}\} \subseteq A_{i^{\underline{m}}}$ so that $\tilde{C}$ does not affect the welfare of $i^{\underline{m}}$. Finally, if $\tilde{m} = \overline{m}$ and $i^{\overline{m}} < \hat{j}^*$, then the third part of Claim 7 implies that either $\{o^{\tilde{m}-1}, p^{\tilde{l}}\} \subseteq \Omega_{i^{\overline{m}}} \setminus A_{i^{\overline{m}}}$ or $\{o^{\tilde{m}-1}, p^{\tilde{l}}\} \subseteq A_{i^{\overline{m}}}$ so that $\tilde{C}$ does not affect the welfare of $i^{\overline{m}}$. Combining all of these observations, we obtain that $\tilde{C}$ is a CIC of μ for $\hat{j}^*$ at $\hat{A}$ as claimed, which is impossible.

If $j^{\hat{l}} \notin I(C) \setminus \{j^*\}$ and $p^{\hat{l}} \in O(C)$, let $\hat{m}$ be such that $o^{\hat{m}-1} = p^{\hat{l}}$ and consider

$$\tilde{C}' \equiv (i^{\hat{m}}, o^{\hat{m}}, \dots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \dots, j^{\hat{l}}, p^{\hat{l}}).$$

As when $j^{\hat{l}} \in I(C) \setminus \{j^*\}$, $\tilde{C}'$ is a cycle of μ.

Next, we show that $\hat{m} \geq \underline{m}$. Note that we must have $\hat{l} < L$ since $p^{\hat{l}} \in O(C)$ and, given that C increases the welfare of $\hat{j}^* = j^1$, $p^L \notin O(C)$. Since $\{j^{\tilde{l}+1}, \ldots, j^L\} \cap \{i^{m'+1}, \ldots, i^M\} = \emptyset$ (by definition of $\tilde{l}$), $\hat{l} \geq \tilde{l} + 1$, and $\hat{l} < L$, we must have $p^{\hat{l}} \in \{o^1, \ldots, o^{m'-1}\}$. Since j^* exchanges two undesirable objects in $\hat{C}$, we must have $o^1 \notin O(\hat{C})$ and thus $p^{\hat{l}} \in \{o^2, \ldots, o^{m'-1}\}$. Since $\hat{l} < L$, this last finding implies $j^{\hat{l}+1} = i^{\hat{m}-1} \neq j^*$. Hence, $\hat{m} - 1 \geq \underline{m}$.

Given that $\hat{m} \geq \underline{m}$, the remainder of the argument is exactly as when $j^{\hat{l}} \in I(C) \setminus \{j^*\}$.

Finally, in case of $\hat{l} = L$ and $j^L \notin I(C)$ an analog of our earlier arguments shows that

$$(i^{m'}, o^{m'}, \ldots, i^{\tilde{m}-1}, o^{\tilde{m}-1}, j^{\tilde{l}}, p^{\tilde{l}}, \ldots, j^L, p^L)$$

is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, which is impossible.

Case 1.3: $\boldsymbol{i^{\tilde{m}'} \neq i^1}$. The argument is analogous to that in Case 1.2 with the only difference being that we consider $\tilde{l}'$ and $\tilde{m}'$ instead of $\hat{l}$ and $\hat{m}$. We omit the details.

Case 2: $\boldsymbol{\{i^{m'+1}, \ldots, i^M\} \cap I(\hat{C}) = \emptyset}$. We first argue that $m(\underline{l}) < \underline{m}$. Assume to the contrary that $m(\underline{l}) \geq \underline{m}$ and consider

$$\tilde{C} \equiv (j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{m'-1}, o^{m'-1}).$$

The definitions of $\underline{l}$ and $m(\underline{l})$ imply that $\tilde{C}$ is a cycle of μ. Since $m(\underline{l}) \geq \underline{m} > 1$, we have that $j^* \notin \{i^{m(\underline{l})}, \ldots, i^{m'-1}\}$. Since $\{j^2, \ldots, j^{\underline{l}}\} \cap I(C) = \emptyset$ by the definition of $\underline{l}$, $j^* \in I(\tilde{C})$ is possible only if $j^* = \hat{j}^*$, which we have already ruled out above. Hence, we must have $j^* \notin I(\tilde{C})$. Given that $\underline{m} \leq m(\underline{l}) < m' = \overline{m}$, Claim 7 implies that there does not exist an $m \in \{m(\underline{l}), \ldots, m' - 1\}$ such that $i^m < \hat{j}^*$ and such that C affects i^m. Finally, Claim 6 implies that $o \notin O(\tilde{C})$. Hence, $\tilde{C}$ is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, which contradicts $\mu \in \mathcal{M}^{i-1}(\hat{A})$.

Next, we argue that $j^{\underline{l}+1} = j^*$. If $p^{\underline{l}} \notin O(C)$, we have that $j^{\underline{l}+1} = i^{m(\underline{l})}$ (recall that $\underline{l} < L$ since $p^L \notin O(C)$ given that C increases the welfare of $\hat{j}^*$ in the case we consider here). Hence, we obtain a contradiction to $m(\underline{l}) < \underline{m}$ unless $j^{\underline{l}+1} = j^*$. If $p^{\underline{l}} \in O(C)$, we must have $p^{\underline{l}} = o^{m(\underline{l})-1}$ and $j^{\underline{l}+1} = i^{m(\underline{l})-1}$. If $m(\underline{l}) = 2$, we obtain the desired conclusion that $j^{\underline{l}+1} = j^*$. If $m(\underline{l}) = 1$, we have that $j^{\underline{l}+1} = i^M$. Since $\underline{l} + 1 \in \{2, \ldots, L\}$, we have $j^{\underline{l}+1} \neq \hat{j}^*$. Hence, $M > m'$ and we obtain a contradiction to $\{i^{m'+1}, \ldots, i^M\} \cap I(\hat{C}) = \emptyset$. If $p^{\underline{l}} \in O(C)$ and $m(\underline{l}) \geq 3$, we obtain a contradiction to $m(\underline{l}) < \underline{m}$.

We split the remainder of the proof in Case 3 into two subcases.

Case 2.1: $p^{\underline{l}+1} \in A_{j^*}$. Let $\tilde{l}$ be the first integer in $(\underline{l}+1,\ldots,L)$ such that either $j^{\tilde{l}} \in I(C)\setminus\{j^*\}$ or $p^{\tilde{l}} \in O(C)\cup\{p^L\}$.

If $j^{\tilde{l}} \in I(C)\setminus\{j^*\}$, let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$. If $m^* < \underline{m}$, analogs of our earlier arguments reveal that

$$(j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^{\tilde{l}-1}, p^{\tilde{l}-1}, i^{\tilde{m}}, o^{\tilde{m}}, \ldots, i^M, o^M).$$

is a CIC of μ for j^* at $\hat{A}$, which is impossible. Hence, we must have $m^* \geq \overline{m} = m'$. Now consider again

$$\tilde{C} \equiv (j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}}, i^{m(\underline{l})}, o^{m(\underline{l})}, \ldots, i^{m'-1}, o^{m'-1}).$$

Since j^* trades two desirable objects in $\hat{C}$, Claims 7 and 8 imply that there cannot exist an $m < m'$ such that $i^m < \hat{j}^*$ and such that C decreases the welfare of i^m. But then $\tilde{C}$ is a CIC of μ for $\hat{j}^*$ at A and, given that $m^* \geq m'$, also at $\hat{A}$, which is impossible.

If $j^{\tilde{l}} \notin I(C)\setminus\{j^*\}$ and $p^{\tilde{l}} \in O(C)$, let $\tilde{m}$ be such that $o^{\tilde{m}-1} = p^{\tilde{l}}$. If $\tilde{m}-1 = M$, then

$$(j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^{\tilde{l}}, p^{\tilde{l}})$$

is a CIC of μ for j^* at $\hat{A}$, which is impossible. Hence, we must have $\tilde{m}-1 < M$ and therefore $i^{\tilde{m}} \neq j^*$. As in the case of $j^{\tilde{l}} \in I(C)\setminus\{j^*\}$,

$$(j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^{\tilde{l}}, p^{\tilde{l}}, i^{\tilde{m}}, o^{\tilde{m}}, \ldots, i^M, o^M)$$

is a CIC of μ for j^* at $\hat{A}$, which is impossible, unless $m^* \geq \overline{m} = m'$. However, if $m^* \geq \overline{m}$, the sequence $\tilde{C}$ defined above is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, which is impossible.

Finally, if $\tilde{l} = L$, $j^L \notin I(C)\setminus\{j^*\}$, and $p^L \notin O(C)$ we obtain a similar contradiction by considering

$$(j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^L, p^L, i^{m'}, o^{m'}, \ldots, i^M, o^M).$$

Case 2.2: $p^{\underline{l}} \in \Omega_{j^*}\setminus A_{j^*}$. We first argue that $p^{\underline{l}+1} \neq o^M$. If $p^{\underline{l}+1} = o^M$, then $i^M \in I(\hat{C})$. Since $\{i^{m'+1},\ldots,i^M\}\cap I(\hat{C}) = \emptyset$, we have $M = m'$. But then $\underline{l}+1 = L$ and $p^L \in O(C)$, which is impossible given that C and $\hat{C}$ both increase the welfare of $\hat{j}^*$.

As in Case 2.1, let $\tilde{l}$ be the first integer in $(\underline{l}+1,\ldots,L)$ such that either $j^{\tilde{l}} \in I(C)\setminus\{j^*\}$ or $p^{\tilde{l}} \in O(C)\cup\{p^L\}$.

If $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$, let $\tilde{m}$ be such that $i^{\tilde{m}} = j^{\tilde{l}}$ and consider

$$\tilde{C} \equiv (i^{\tilde{m}}, o^{\tilde{m}}, \ldots, i^M, o^M, j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^{\tilde{l}-1}, p^{\tilde{l}-1}).$$

Note that $\tilde{m} \in \{2, \ldots, m' - 1\}$ given that $\{i^{m'+1}, \ldots, i^M\} \cap I(\hat{C}) = \emptyset$ and $\tilde{l} \in \{2, \ldots, L\}$. Since j^* exchanges two undesirable objects in $\hat{C}$, Claims 7 and 9 imply that there does not exist an $m \in \{\underline{m}+1, \ldots, M\}$ such that $i^m < \hat{j}^*$ and such that C decreases the welfare of i^m. If $\tilde{m} = \underline{m}$ and $i^{\underline{m}} < \hat{j}^*$, then the second part of Claim 7 implies that either $\{p^{\tilde{l}-1}, o^{\tilde{m}}\} \subseteq A_{i^{\underline{m}}}$ or $\{p^{\tilde{l}-1}, o^{\tilde{m}}\} \subseteq \Omega_{i^{\underline{m}}} \setminus A_{i^{\underline{m}}}$. Combining our last two findings, we see that $\tilde{C}$ is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, which is impossible, unless $m^* \geq m'$. However, if $m^* \geq m'$, then

$$(i^1, o^1, \ldots, i^{m'-1}, o^{m'-1}, j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}})$$

is a CIC of μ for j^* at $\hat{A}$, which is impossible.

If $j^{\tilde{l}} \notin I(C) \setminus \{j^*\}$ and $p^{\tilde{l}} \in O(C)$, let $\tilde{m}$ be such that $o^{\tilde{m}-1} = p^{\tilde{l}}$. Since j^* trades two undesirable objects in $\hat{C}$, we must have $o^1 \notin O(\hat{C})$ and therefore $\tilde{m} - 1 \geq 2$. Furthermore, we must have $\tilde{l} < L$ since $p^L \notin O(C)$ and therefore $j^{\tilde{l}+1} = i^{\tilde{m}-1} \neq \hat{j}^*$. Combining our last two findings, we have that $\underline{m} \leq \tilde{m} < m'$ and we can proceed just as in case $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$ to derive a contradiction to $\mu \in \mathcal{M}^{i-1}(\hat{A})$.

Finally, if $j^{\tilde{l}} \notin I(C) \setminus \{j^*\}$ and $\tilde{l} = L$,

$$\tilde{C}' \equiv (i^{m'}, o^{m'}, \ldots, i^M, o^M, j^{\underline{l}+1}, p^{\underline{l}+1}, \ldots, j^L, p^L).$$

a similar argument to that for the case of $j^{\tilde{l}} \in I(C) \setminus \{j^*\}$ shows that $\tilde{C}'$ is a CIC of μ for $\hat{j}^*$ at $\hat{A}$, which is impossible, unless $m^* \geq m'$. However, as before, if $m^* \geq m'$, then

$$(i^1, o^1, \ldots, i^{m'-1}, o^{m'-1}, j^1, p^1, \ldots, j^{\underline{l}}, p^{\underline{l}})$$

is a CIC of μ for j^* at $\hat{A}$, which is impossible.

□

B Proofs from Section 6

Proof of Proposition 2. The only difference between our original model and the minimum guarantees and unendowed objects variant is that i may receive sets containing more than $|\Omega_i|$. Without the r extra objects, this difference is moot. Yet, it is possible to embed matching problems with unendowed objects in our original model. We associate each agent $i \in I$ with r "dummy objects" $\Omega_i^\delta \equiv \{\delta_{i1}, \ldots, \delta_{ir}\}$. Let $O^\delta \equiv \cup_{i \in I} \Omega_i^\delta$ be the set of all dummy objects and let $\tilde{\Omega}_i = \Omega_i \cup \Omega_i^\delta$. We define a dummy agent d endowed with the extra objects, i.e. $\tilde{\Omega}_d = O \setminus (\cup_{i \in I} \Omega_i)$. However we let d find all objects, including dummy ones, to be desirable, i.e. $A_d = O \cup O^\delta$. Consider now the matching problem (A, A_d) with trichotomous preferences involving agents $I \cup \{d\}$ with endowments $(\tilde{\Omega}, \tilde{\Omega}_d)$.

To complete the proof, we show that if μ is an individually rational and Pareto-efficient matching for this augmented economy, then it corresponds to an individually rational and Pareto-efficient matching for the original economy with the r extra. Thus, even for minimum guarantees and unendowed objects, IRP mechanisms satisfy the three axioms that we seek.

Let μ be a matching for the augmented economy. Define $\nu : I \to 2^O$ such that for each $i \in I$, $\nu(i) = \mu(i) \cap O$. Since for each pair of agents $i, j \in I$, $\mu(i) \cap \mu(j) = \emptyset$, $\nu(i) \cap \nu(j) = \emptyset$ as well.

Now suppose that μ is individually rational for the augmented economy. Then, for each $i \in I$, $\mu(i) \subseteq \tilde{\Omega}_i \cup A_i$ and $|\mu(i)| = |\tilde{\Omega}_i| = |\Omega_i| + r$. The former fact implies that $\nu(i) \subseteq \Omega_i \cup A_i$. The latter fact, along with $|O \setminus (\cup_{i \in I} \Omega_i)\,| = r$, implies that $|\nu(i)| = |\mu(i) \cap O| \geq |\Omega_i|$. Thus, in the original economy, $\nu(i) \succsim_i \Omega_i$, so ν is not only a matching but it is individually rational.

Suppose that μ is also Pareto-efficient for the augmented economy. If ν is not Pareto-efficient for the original economy, then there is $\nu' \in \mathcal{M}$ such that for each $i \in I$, $\nu'(i) \succsim_i \nu(i)$ and for some $i \in I$, $\nu'(i) \succ_i \nu(i)$. For each $i \in I$, let $r_i \equiv |\nu'(i)| - |\Omega_i|$ be the number of additional objects that i receives relative to her endowment. Let $\mu' : I \cup \{d\} \to 2^{O \cup O^\delta}$ be such that

1. for each $i \in I$, $\nu'(i) \subseteq \mu'(i) \subseteq \tilde{\Omega}_i \cup A_i$ and $|\mu'(i)| = |\Omega_i| + r_i = |\tilde{\Omega}_i|$, and
2. $\mu'(d) = \cup_{i \in I}(\Omega_i^\delta \setminus \mu'(i))$.

Then, for each $i \in I, \mu'(i) \setminus \nu'(i) \subseteq \Omega_i^\delta$ and μ' is a matching for the augmented economy. Since $A_d = O \cup O^\delta$, in the augmented economy we have $\mu'(d) \sim_d \mu(d) \sim_d \tilde{\Omega}_d$. Furthermore, since ν' Pareto-improves ν in the orignal economy, and d is indifferent between μ and μ' in the augmented economy, μ' Pareto-improves μ. This contradicts the assumption that μ is Pareto-efficient for the augmented economy. $\square$

Proof of Proposition 3. We first show incompatibility for the case of three agents, two of whom are endowed with a single object while the third of whom is endowed with two objects. So, let $I \equiv \{1, 2, 3\}$ and $O \equiv \{\omega_1, \omega_2, \omega_{31}, \omega_{32}\}$. Let $\Omega_1 \equiv \{\omega_1\}, \Omega_2 \equiv \{\omega_2\}$, and $\Omega_3 \equiv \{\omega_{31}, \omega_{32}\}$.

Suppose that $\succsim$ is a minimally non-trichotomous preference profile such that $A_1 = A_2 = \{\omega_{31}, \omega_{32}\}, a_1 = a_2 = \omega_{31}, A_3 = \{\omega_1, \omega_2\}$, and $a_3 = \omega_2$.

A matching $\mu \in \mathcal{M}$ is individually rational if and only if $\omega_2 \notin \mu(1)$ and $\omega_1 \notin \mu(2)$. Pareto-efficiency is then easily seen to require that μ can be individually rational and Pareto-efficient only if $\mu(3) = \{\omega_1, \omega_2\}$. Hence, there are two Pareto-efficient and individually rational matchings: μ^1 such that $\mu^1(1) = \{\omega_{31}\}, \mu^1(2) = \{\omega_{32}\}$, and $\mu^1(3) = \{\omega_1, \omega_2\}$ and μ^2 such that $\mu^2(1) = \{\omega_{32}\}, \mu^2(2) = \{\omega_{31}\}$, and $\mu^2(3) = \{\omega_1, \omega_2\}$.

Suppose now that φ is strategy-proof, Pareto-efficient, and individually rational.

Case 1: $\boldsymbol{\varphi(\succsim) = \mu^1}$. Consider the trichotomous preference relation $\succsim_2'$ for agent 2 such that $A_2' = \{\omega_{31}\}$. At $(\succsim_1, \succsim_2', \succsim_3)$, a matching $\mu \in \mathcal{M}$ is individually rational if and only if $\omega_2 \notin \mu(1)$, $\omega_1 \notin \mu(2)$, and $\omega_{32} \notin \mu(2)$. There are two Pareto-efficient and individually rational matchings at $(\succsim_1, \succsim_2', \succsim_3)$: ν' such that $\nu'(1) = \{\omega_{31}\}, \nu'(2) = \{\omega_2\}$, and $\nu'(3) = \{\omega_1, \omega_{32}\}$ and μ^2 defined above.

If $\varphi(\succsim_1, \succsim_2', \succsim_3) = \mu^2$, then agent 2 has an incentive to report $\succsim_2'$ when the true preference profile is $\succsim$. Hence, strategy-proofness implies that $\varphi(\succsim_1, \succsim_2', \succsim_3) = \nu'$.

Now, consider the trichotomous preference relation $\succsim_3'$ for agent 3 such that $A_3 = \{\omega_2\}$. At $(\succsim_1, \succsim_2', \succsim_3')$ a matching $\mu \in \mathcal{M}$ is individually rational if and only if $\omega_2 \notin \mu(1), \omega_1 \notin \mu(2), \omega_{32} \notin \mu(2)$, and $\omega_1 \notin \mu(3)$. Thus, $\mu(1) = \{\omega_1\}$. So the unique individually rational and Pareto-efficient matching is λ' such that $\lambda'(1) = \{\omega_1\}, \lambda'(2) = \{\omega_{31}\}$, and $\lambda'(3) = \{\omega_2, \omega_{32}\}$. Therefore, $\varphi(\succsim_1, \succsim_2', \succsim_3') = \lambda'$. However, this contradicts the strategy-proofness of φ since agent 3 has an incentive to report $\succsim_3'$ when the true preference profile is $(\succsim_1, \succsim_2', \succsim_3)$.

Case 2: $\boldsymbol{\varphi(\succsim) = \mu^2}$. Consider the trichotomous preference relation $\hat{\succsim}_1$ for agent 1 such that $\hat{A}_1 = \{\omega_{31}\}$. At $(\hat{\succsim}_1, \succsim_2, \succsim_3)$, a matching $\mu \in \mathcal{M}$ is individually rational if and only if $\omega_2 \notin \mu(1), \omega_{32} \notin \mu(1)$, and $\omega_1 \notin \mu(2)$. Thus, there are two individually rational and Pareto-efficient matchings at $(\hat{\succsim}_1, \succsim_2, \succsim_3)$: $\hat{\nu}$ such that $\hat{\nu}(1) = \{\omega_1\}, \hat{\nu}(2) = \{\omega_{31}\}$, and $\hat{\nu}(3) = \{\omega_2, \omega_{32}\}$ and μ^1 defined above.

If $\varphi(\hat{\succsim}_1, \succsim_2, \succsim_3) = \mu^1$, then agent 1 has an incentive to report $\hat{\succsim}_1$ when the true preference profile is $\succsim$. Hence, strategy-proofness implies $\varphi(\hat{\succsim}_1, \succsim_2, \succsim_3) = \hat{\nu}$.

Consider the minimally non-trichotomous preference relation $\hat{\succsim}_3$ for agent 3 such that $A_3 = \{\omega_1, \omega_2\}$ and $a_3 = \{\omega_1\}$. The individually rational and Pareto-efficient matchings at $(\hat{\succsim}_1, \succsim_2, \hat{\succsim}_3)$ are, again, μ^1 and $\hat{\nu}$. By strategy-proofness $\varphi(\hat{\succsim}_1, \succsim_2, \hat{\succsim}_3) = \hat{\nu}$. Otherwise, $\varphi(\hat{\succsim}_1, \succsim_2, \hat{\succsim}_3) = \mu^1$ and agent 3 has an incentive to report $\hat{\succsim}_3$ when the true preference profile is $(\hat{\succsim}_1, \succsim_2, \succsim_3)$.

Finally, consider the trichotomous preference relation $\bar{\succsim}_3$ for agent 3 such that $\bar{A}_3 = \{\omega_1\}$. At $(\hat{\succsim}_1, \succsim_2, \bar{\succsim}_3)$, a matching $\mu \in \mathcal{M}$ is individually rational if and only if $\omega_2 \notin \mu(1), \omega_{32} \notin \mu(1), \omega_1 \notin \mu(2)$, and $\omega_2 \notin \mu(3)$. Thus, $\mu(2) = \{\omega_2\}$. So the unique individually rational and Pareto-efficient matching is $\hat{\lambda}$ such that $\hat{\lambda}(1) = \{\omega_{31}\}, \hat{\lambda}(2) = \{\omega_2\}$, and $\hat{\lambda}(3) = \{\omega_1, \omega_{32}\}$.

This contradicts the strategy-proofness of φ since agent 3 has an incentive to report $\bar{\succsim}_3$ when the true preference profile is $(\hat{\succsim}_1, \succsim_2, \hat{\succsim}_3)$.

Now we consider the case where the three agents are endowed with more objects. Suppose $\Omega_1 = \{\omega_{11}, \omega_{12}, \dots, \omega_{1k_1}\}$, $\Omega_2 = \{\omega_{21}, \omega_{22}, \dots, \omega_{2k_2}\}$, and $\Omega_2 = \{\omega_{31}, \omega_{32}, \dots, \omega_{3k_3}\}$ such that $k_3 \geq 2$. We now require that for each $i \in I$, and each "extra" objects $o \in \{\omega_{12}, \dots, \omega_{1k_1}\} \cup \{\omega_{22}, \dots, \omega_{2k_2}\} \cup \{\omega_{33}, \dots, \omega_{3k_3}\}$, i does not consider o to be desirable, i.e. $o \notin A_i$. Individual rationality requires that 1 retains the objects $\omega_{12}, \dots, \omega_{1k_1}$, 2 retains the objects $\omega_{22}, \dots, \omega_{2k_2}$, and 3 retains the objects $\omega_{33}, \dots, \omega_{3k_3}$. The remainder of the proof is exactly as above.

Finally, we consider the case where there are more than three agents. Let $1, 2,$ and 3 be three of the agents in I such that $|\Omega_3| \geq 2$. For each $i \in I \setminus \{1, 2, 3\}$, we consider only preferences such that $A_i = \Omega_i$ and for each $o \in \Omega_i$, there is no $j \in I$ such that $o \in A_j$. That is, agents other than 1, 2, and 3 are degenerate in the sense that individual rationality requires that they consume their endowments. Thus, the proof proceeds as above. □

Proof of Proposition 4. Let A be a profile of desirable sets and k be a profile of targets. Let $i \in I$. By definition of all-or-nothing preferences, it is impossible for i to gain by reporting a distinct target $\hat{k}_i \neq k_i$. Thus, we show that i does not benefit by reporting an alternative desirable set $\hat{A}_i \neq A_i$.

If $\hat{A}_i$ is a *profitable* manipulation for i, then $|\Omega_i \cap A_i| < k_i$, $\mu(i) = \Omega_i$ for each $\mu \in \mathcal{M}^i(A, k)$, and there has to exist a $\hat{\mu} \in \mathcal{M}^i(\hat{A}_i, A)$ such that $|\hat{\mu}(i) \cap A_i| = k_i$ and $\hat{\mu}(i) \subseteq \Omega_i \cup A_i$.

Suppose there is $t < i$ such that $K^t(A, k) \neq K^t((\hat{A}_i, A_{-i}), k)$. Let t be the first of such agents. Since for each $t' < t$, $K^{t'}(A, k) = K^{t'}((\hat{A}_i, A_{-i}), k)$, we draw the following two conclusions:

1. Since $\mu(i) = \Omega_i$ and $\mu \in \mathcal{M}^{t-1}(A, k)$, the definition of t implies $\mu \in \mathcal{M}^{t-1}((\hat{A}_i, A_{-i}), k)$. This, in turn, implies that
$$K^t((\hat{A}_i, A_{-i}), k) \geq |\mu(t) \cap A_t| = K^t(A, k).$$

2. Since $|\hat{\mu}(i) \cap A_i| = k_i$, the definition of t implies $\hat{\mu} \in \mathcal{M}^{t-1}(A, k)$. This, in turn, implies that
$$K^t(A, k) \geq |\hat{\mu}(t) \cap A_t| = K^t((\hat{A}_i, A_{-i}), k).$$

Together, we have a contradiction of the supposition that $K^t(A, k) \neq K^t((\hat{A}_i, A_{-i}), k)$. Thus, for each $t < i$, $K^t(A, k) = K^t(\hat{A}_i, A_{-i})$, implying that $\hat{\mu} \in \mathcal{M}^{i-1}(A, k)$. So $K^i(A, k) \geq K^i((\hat{A}_i, A_{-i}), k)$, contradicting the assumption that $\hat{A}_i$ is a profitable manipulation for i. □

Proof of Proposition 5. We show this incompatibility for the case of three agents, each of whom is endowed with two objects. So, let $I \equiv \{1, 2, 3\}$ and $O \equiv \{\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22}, \omega_{31}, \omega_{32}\}$. Let $\Omega_1 \equiv \{\omega_{11}, \omega_{12}\}$, $\Omega_2 \equiv \{\omega_{21}, \omega_{22}\}$, and $\Omega_3 \equiv \{\omega_{31}, \omega_{32}\}$. We fix the desirable sets and prove the impossibility by considering only manipulations of the lower bounds. So, let $A_1 \equiv \{\omega_{21}, \omega_{31}, \omega_{32}\}$, $A_2 \equiv \{\omega_{11}, \omega_{12}\}$, and $A_3 \equiv \{\omega_{11}, \omega_{12}, \omega_{21}\}$. We start with the case where, for each $i \in I$, $\underline{k}_i = 1$.

There are four individually rational and Pareto-efficient matchings: μ^1 such that $\mu^1(1) = \{\omega_{21}, \omega_{31}\}$, $\mu^1(2) = \{\omega_{11}, \omega_{12}\}$, and $\mu^3(3) = \{\omega_{22}, \omega_{32}\}$; μ^2 such that $\mu^2(1) = \{\omega_{21}, \omega_{32}\}$, $\mu^2(2) = \{\omega_{11}, \omega_{12}\}$, and $\mu^2(3) = \{\omega_{22}, \omega_{31}\}$; μ^3 such that $\mu^3(1) = \{\omega_{31}, \omega_{32}\}$, $\mu^3(2) = \{\omega_{11}, \omega_{21}\}$, and $\mu^3(3) = \{\omega_{12}, \omega_{22}\}$; and μ^4 such that $\mu^4(1) = \{\omega_{31}, \omega_{32}\}$, $\mu^4(2) = \{\omega_{12}, \omega_{21}\}$, and $\mu^4(3) = \{\omega_{11}, \omega_{22}\}$.

Next, consider the lower bound $\underline{k}'_2 = 2$ for agent 2. At the profile of lower bounds, $(\underline{k}_1, \underline{k}'_2, \underline{k}_3)$, μ^1 and μ^2 are the only individually rational and Pareto-efficient matchings.

Finally, consider the lower bound $\underline{k}'_3 = 2$ for agent 3. At the profile of lower bounds, $(\underline{k}_1, \underline{k}_2, \underline{k}'_3)$, μ^3 and μ^4 are the only individually rational and Pareto-efficient matchings.

If an individually rational and Pareto-efficient mechanism selects μ^1 or μ^2 for the profile of lower bounds $\underline{k}$, then agent 3 benefits by reporting the lower bound $\underline{k}'_3$. On the other hand, if it selects μ^3 or μ^4, then agent 2 benefits by reporting $\underline{k}'_2$. Thus, regardless of the choice made at $\underline{k}$, the mechanism is not strategy-proof. $\square$