Rethinking efficiency in random allocation

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Efficiency in Economics

Efficiency in Economics

Central normative criterion for over a century

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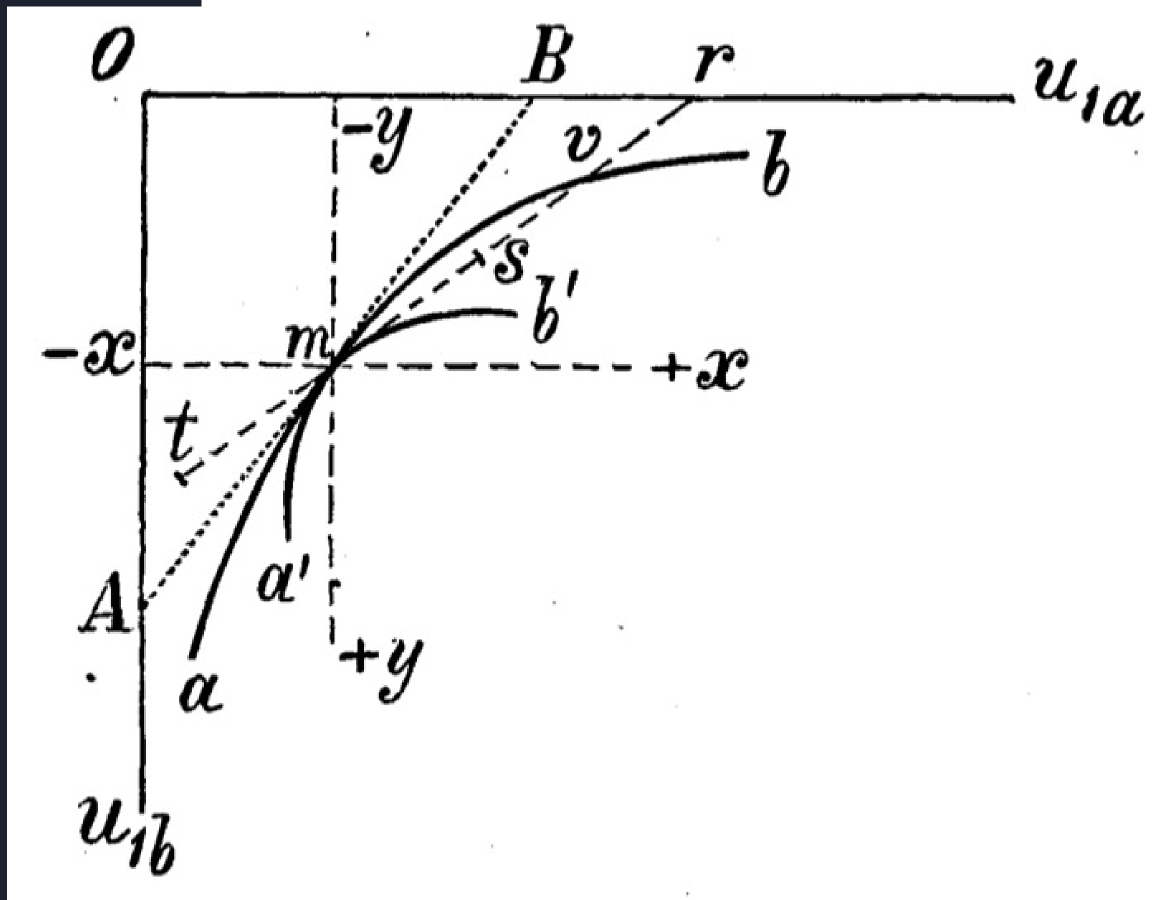
Central normative criterion for over a century

First reference by Pareto in 1902

Il punto m gode di tale proprietà perchè, essendo punto di equilibrio, le due curve di indifferenza sono tangenti. Se m è un punto ove due linee di indifferenza si tagliano, è manifesto che, se uno degli individui è in s e l'altro in t, sulla retta $s\,m$, per modo che si abbia sempre



Efficiency in Economics

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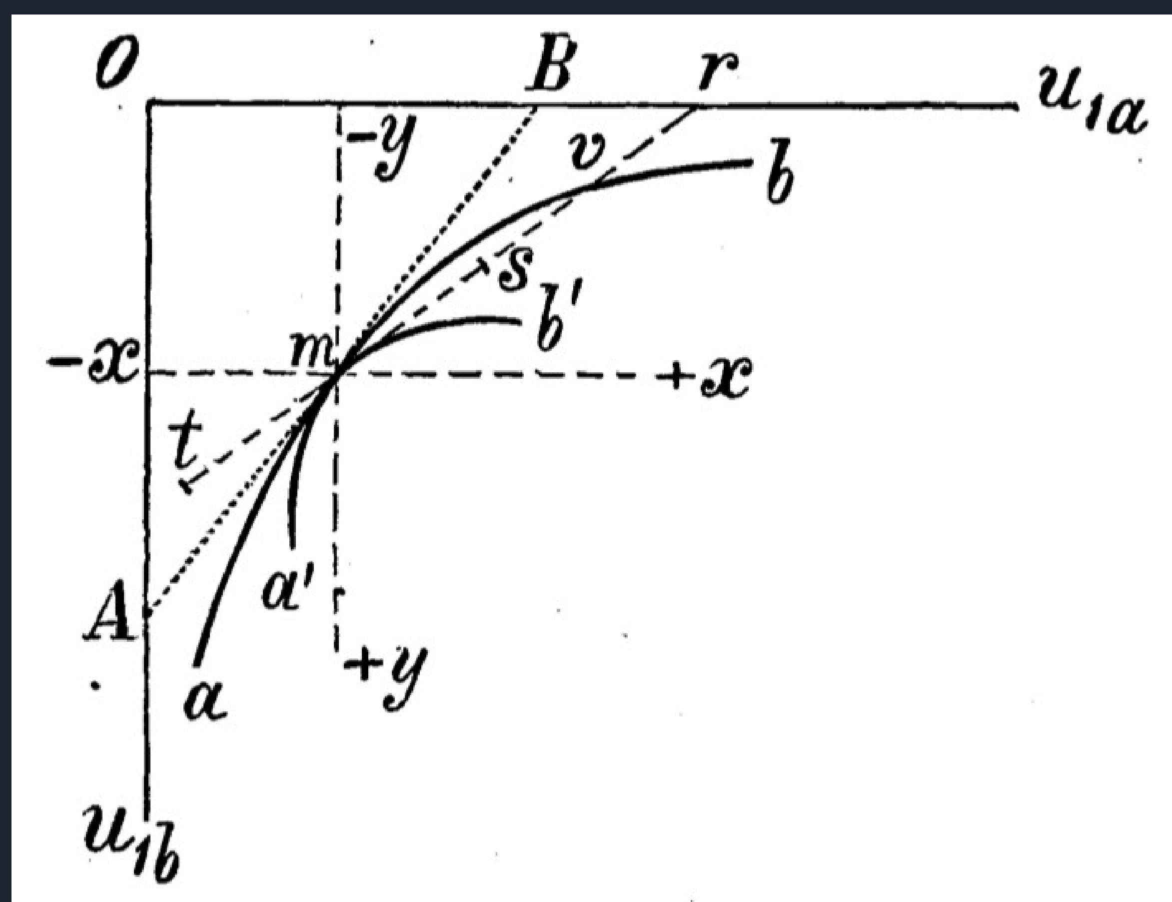

"The point m enjoys the property that it is not possible in departing from it, by barter, or by similar arrangement such that what is taken away from one individual is given to another, to increase the total ophelimities of both individuals."

Efficiency in Economics

Appears in Edgeworth (1881) even earlier

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In general let there be m contractors and n subjects of contract, n variables. Then by the principle (3) [above, p. 23] the state of equilibrium may be considered as such that the utility of any one contractor must be a maximum *relative to* the utilities of the other contractors being constant, or not decreasing; which

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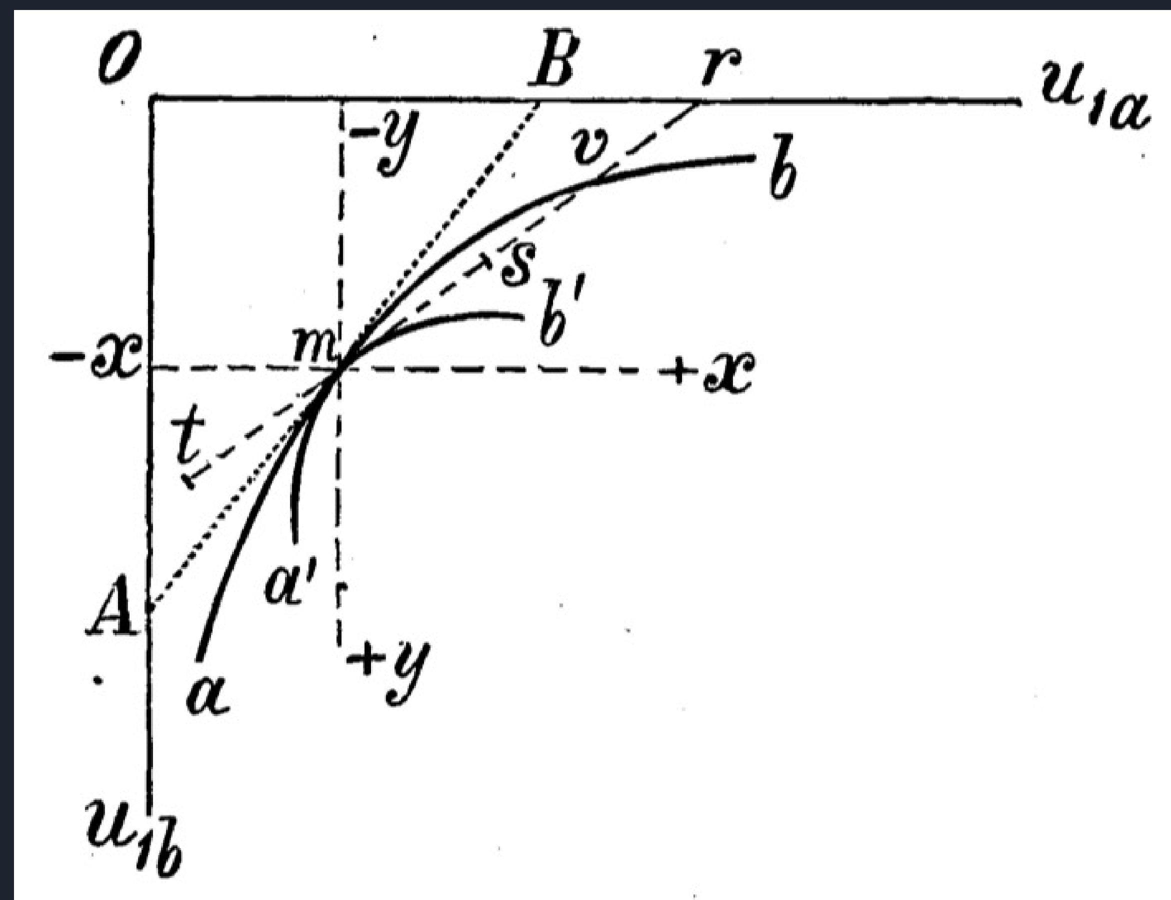
[Should we have called it Edgeworth-efficiency?]

A Minor Digression (on that parenthetical note)

It would appear that Pareto didn't know about the Edgeworth box in 1902



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He actually discovered it a few years later

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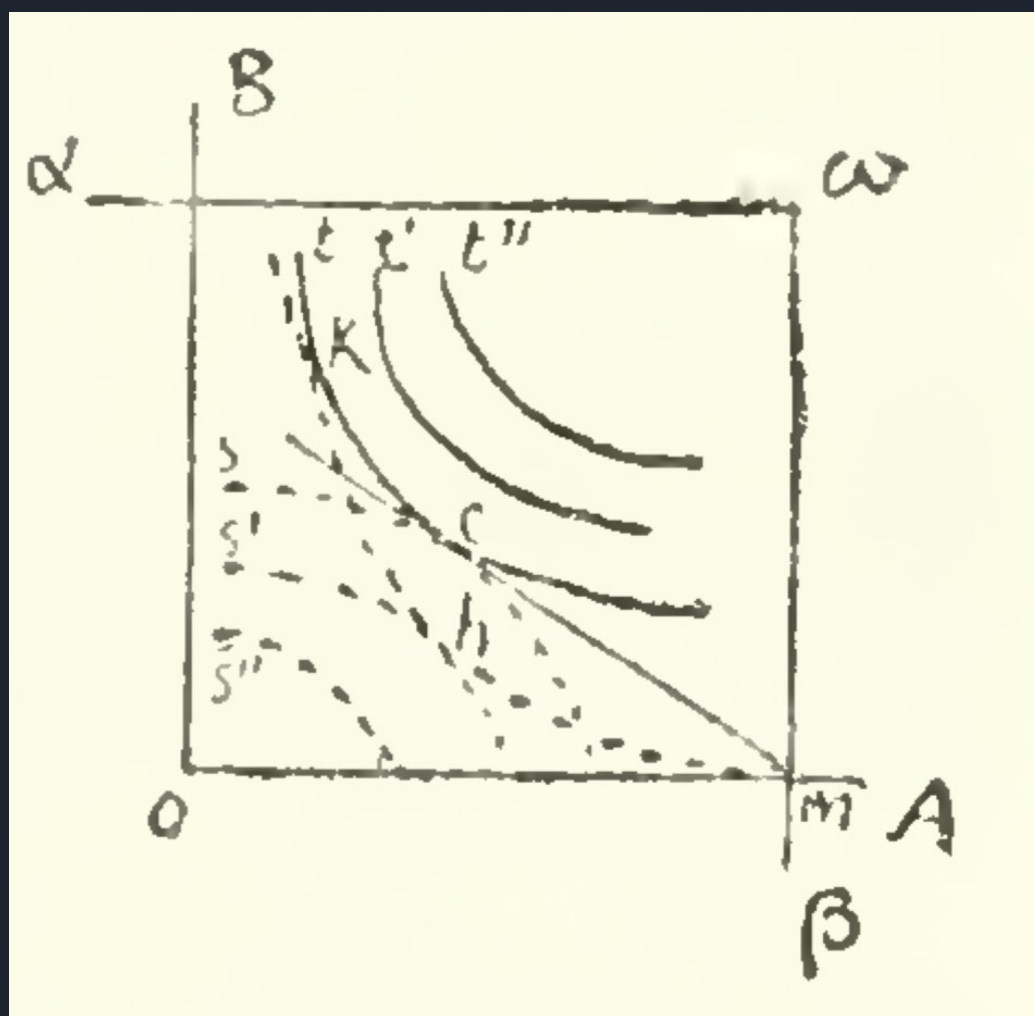

Pareto (1906)

A Minor Digression (on that parenthetical note)

But didn't Edgeworth invent it?

A Minor Digression (on that parenthetical note)

This is Edgeworth's "Edgeworth box"

FIG. 1.

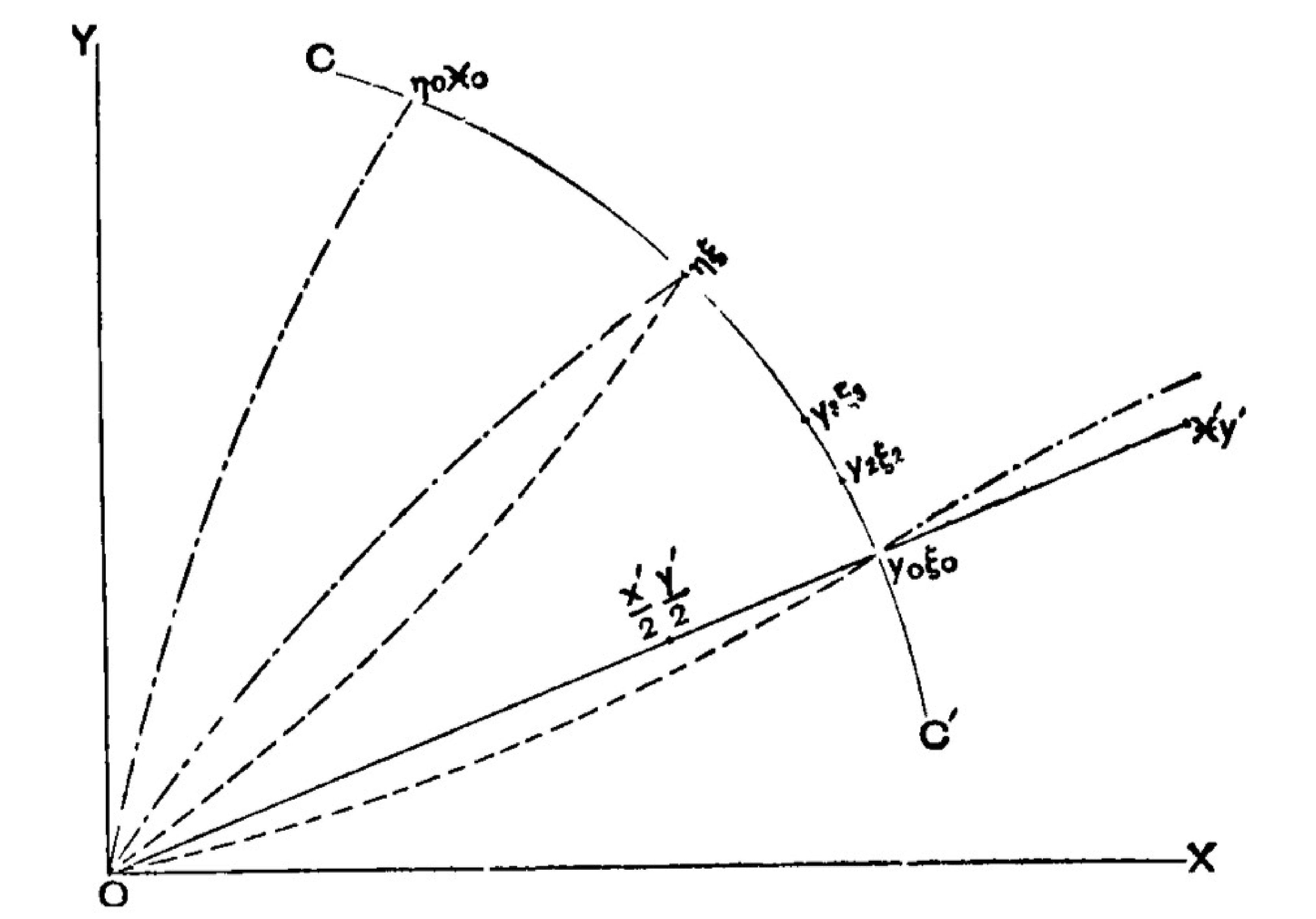

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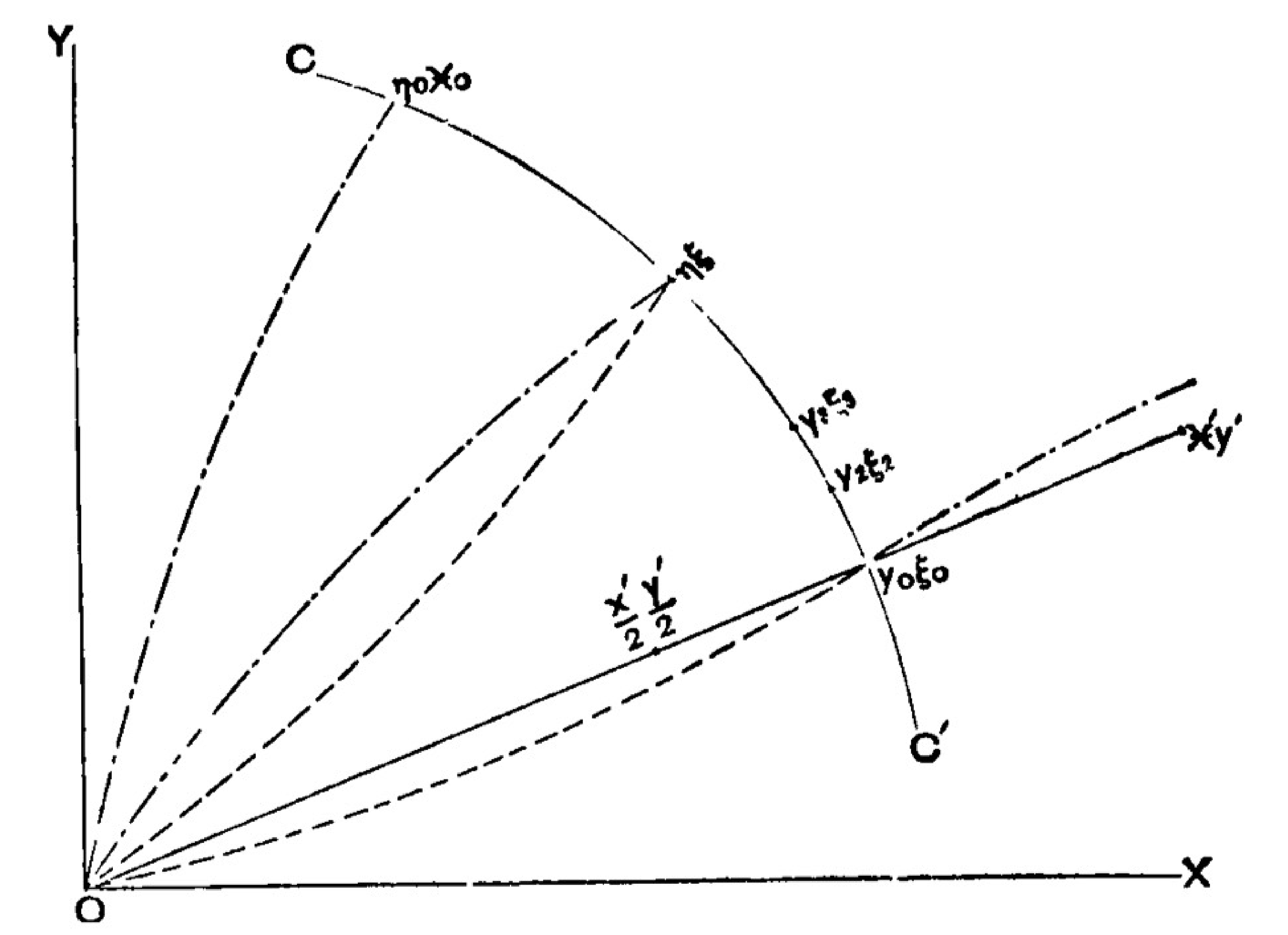
FIG. 1.



It depicts the contract curve (CC') in his Robinson Crusoe economy

A Minor Digression (on that parenthetical note)

Should we switch to

the Pareto-box

and

Edgeworth-efficiency?

A Minor Digression (on that parenthetical note)

Should we switch to the Pareto-box and Edgeworth-efficiency?

End of digression

Efficiency in Economics

Edgeworth's ~~Pareto's~~ formulation of efficiency works well for divisible goods where we look for tangency

Efficiency in Economics

Pareto's formulation of efficiency works well for divisible goods where we look for tangency

Arrow (1951) brings this principle to a setting where that's not possible

Efficiency in Economics

In response to the debate over the compensation principle he argues that the comparison should be between a state x and a state y *after* compensations are made

i.e. what we now all know as Pareto-efficiency

Efficiency in Economics

Notably, he spends at least a few lines "justifying" it

This formulation is certainly not debatable except perhaps on a philosophy of systematically denying people whatever they want. Actually, it is a rather roundabout way of saying something simple. For

Efficiency in Economics

By the 70s this is generally taken for granted

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Zeckhauser (1973) lists it as one of two criteria and simply says Pareto-optimality is "unambiguous"

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Acheiving Pareto-efficiency in random allocation is the entire point of Hylland & Zeckhauser (1979)

Random Allocation

We study the same kind of random allocation problem

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Departure from H & Z (1973) : Ordinality

Random Allocation

We study the same kind of random allocation problem

Departure from H & Z (1973): Ordinality

Literature on ordinal random allocation starts with Bogomolnaia & Moulin (2001)

Random Allocation

Is the right formulation of efficiency still "unambiguous"?

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Is the right formulation of efficiency still "unambiguous"?

We don't think so

Random Allocation

Is the right formulation of efficiency still "unambiguous"?

We don't think so

But broad acceptance of Pareto-efficiency as self-evidently appropriate might be why the current definition hasn't seen any discussion

Outline of the Talk

- ☆ The random allocation model

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- A characterization of efficient, strategy-proof, ... rules

Random Allocation

$N \leftarrow$ set of agents

Random Allocation

N $\qquad\qquad$ A

set of objects

Random Allocation

$$|N| = |A|$$

Agents have vNM preferences

u_{ia} ← i's utility from object a

Random Allocation

$$|N| = |A|$$

Agents have vNM preferences

u_{ia}

$\mathcal{U} \leftarrow$ all utility vectors without ties

Random Allocation

$$|N| = |A|$$

Agents have vNM preferences

u_{ia}

$\underline{\mathcal{U}}$ In the paper we allow any superset of this domain

(Richer set of preferences makes most of our results easier)

Random Allocation

$$|N| = |A|$$

Agents have vNM preferences

u_{ia}

$\mathcal{U}$

Each i gets $\pi_i \leftarrow$ probability distribution over A

Random Allocation

$$|N| = |A|$$

Agents have vNM preferences

u_{ia}

$\mathcal{U}$

Each i gets π_i

$u_i \cdot \pi_i$ ← i's expected utility from π_i

Random Allocation

$\Pi \leftarrow$ bistochastic matrices

Random Allocation

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There's always a lottery of matchings that induces the marginals given by a bistochastic matrix

(Birkhoff (1946), von Neumann (1953))

Random Allocation

Π

$\varphi : \mathcal{U}^N \longrightarrow \Pi$

(Allocation) rule maps utility profiles to allocations

Properties of Allocations/Rules (Axioms)

π is efficient if

$\nexists \pi' \in \Pi$ such that $\forall i \; u_i \cdot \pi_i' \geq u_i \cdot \pi_i$

$\exists i \; u_i \cdot \pi_i' > u_i \cdot \pi_i$

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The same notion of efficiency that dates back to Edgeworth

Properties of Allocations/Rules (Axioms)

π is fair (no-envy) if

$\nexists\ i, j \in N$ such that

$$u_i \cdot \pi_j > u_i \cdot \pi_i$$

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Dates back to Tinbergen (1930)

Properties of Allocations/Rules (Axioms)

ϕ is symmetric if $\forall u \in \mathcal{U}^N$ and $\forall i, j \in N$

$$u_i = u_j \implies \phi_i(u) = \phi_j(u)$$

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Aristotle (350 BCE?)

Properties of Allocations/Rules (Axioms)

ϕ is strategy-proof if $\forall u \in \mathcal{U}^N, \forall i \in N, \forall u_i' \in \mathcal{U}$

$$\underset{T}{u_i} \cdot \underset{T}{\phi_i(u)} \geq \underset{T}{u_i} \cdot \underset{L}{\phi_i(u_i', u_{-i})}$$

Properties of Allocations/Rules (Axioms)

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Black (1948) & Farquharson (1956)

Properties of Allocations/Rules (Axioms)

ϕ is non-bossy if $\forall u \in \mathcal{U}^N, \forall i \in N, \forall u_i' \in \mathcal{U}$

$$\phi_i(u) = \phi_i(u_i', u_{-i})$$

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Satterthwaite & Sonnenschein (1981)

Properties of Allocations/Rules (Axioms)

ϕ is continuous if $\phi(u)$ is continuous in u

Properties of Allocations/Rules (Axioms)

ϕ is continuous if $\phi(u)$ is continuous in u

Chichilnisky (1980)

Properties of Allocations/Rules (Axioms)

$\mathcal{P} \leftarrow$ linear orders over A

Properties of Allocations/Rules (Axioms)

$\mathcal{P}$

$\forall P \in \mathcal{P}$ $\quad u^P \leftarrow$ vNM utilities consistent with P

Properties of Allocations/Rules (Axioms)

$\mathcal{P}$

$\forall\, P \in \mathcal{P}$ $\quad u^P \leftarrow$ vNM utilities consistent with P

i.e. u such that

$$u_a > u_b \iff a\ P\ b$$

Properties of Allocations/Rules (Axioms)

ϕ is Ordinal if $\forall\, u, u' \in \mathcal{U}^N$

u & u' consistent with same $P \in \mathcal{P}^N$

$$\Downarrow$$

$$\phi(u) = \phi(u')$$

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Gibbard (1977) ; Bogomolnaia & Moulin (2001)

Why Ordinality?

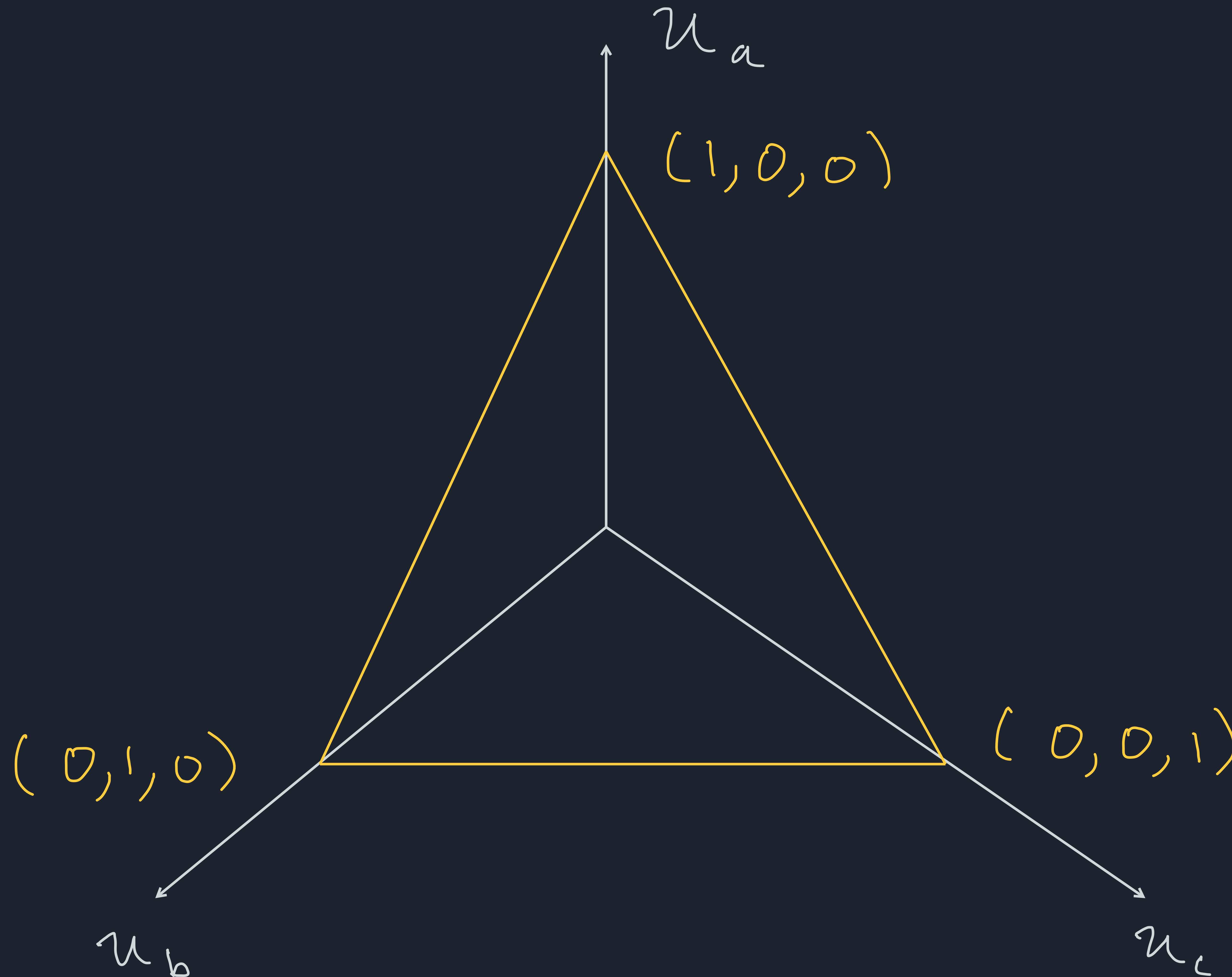
I'll offer two justifications

Why Ordinality?

Bogomolnaia & Moulin's (2001) argument:

the central assumption in this paper.[5] It can be justified by the limited rationality of the agents participating in the mechanism. There is convincing experimental evidence that the representation of preferences over uncertain outcomes by VNM utility functions is inadequate (see, e.g., Kagel and Roth [11]). One interpretation of this literature is that the formulation of rational preferences over a given set of lotteries is a complex process that most agents do not engage into if they can avoid it. An ordinal mechanism allows the participants to formulate only this part of their preferences that does not require to think about the choice over lotteries. It is genuinely simpler to implement an ordinal mechanism than a cardinal one.

Why Ordinality?



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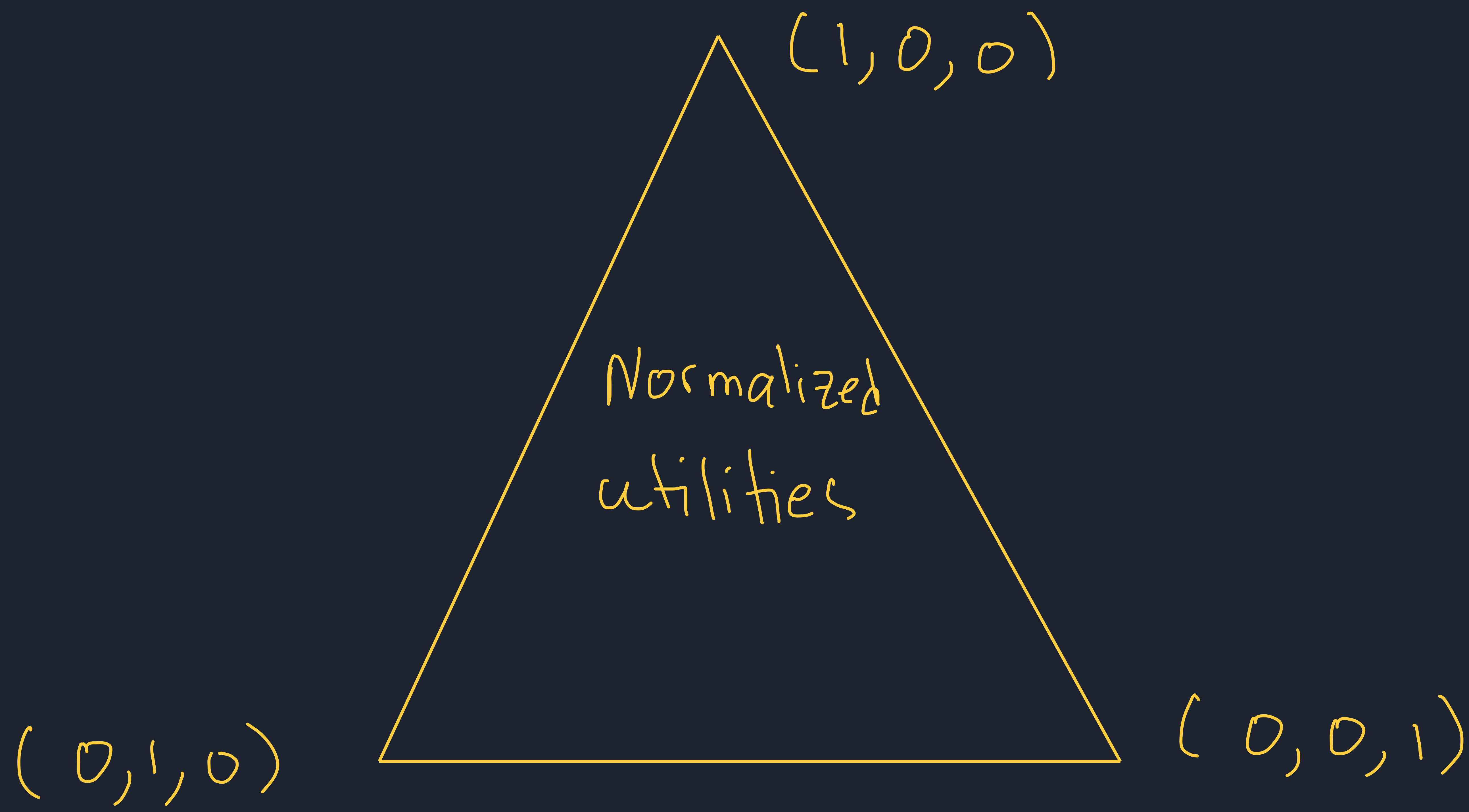

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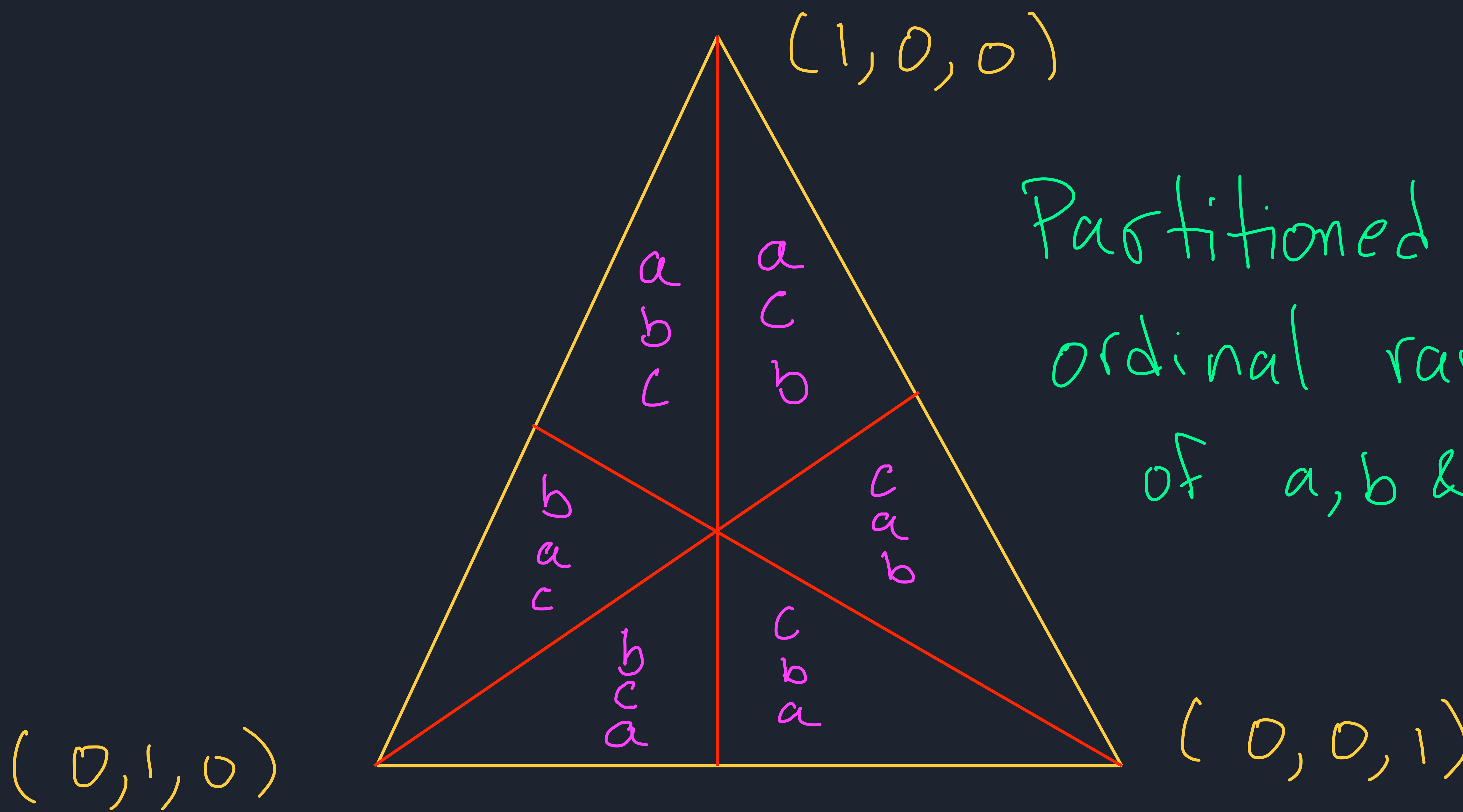
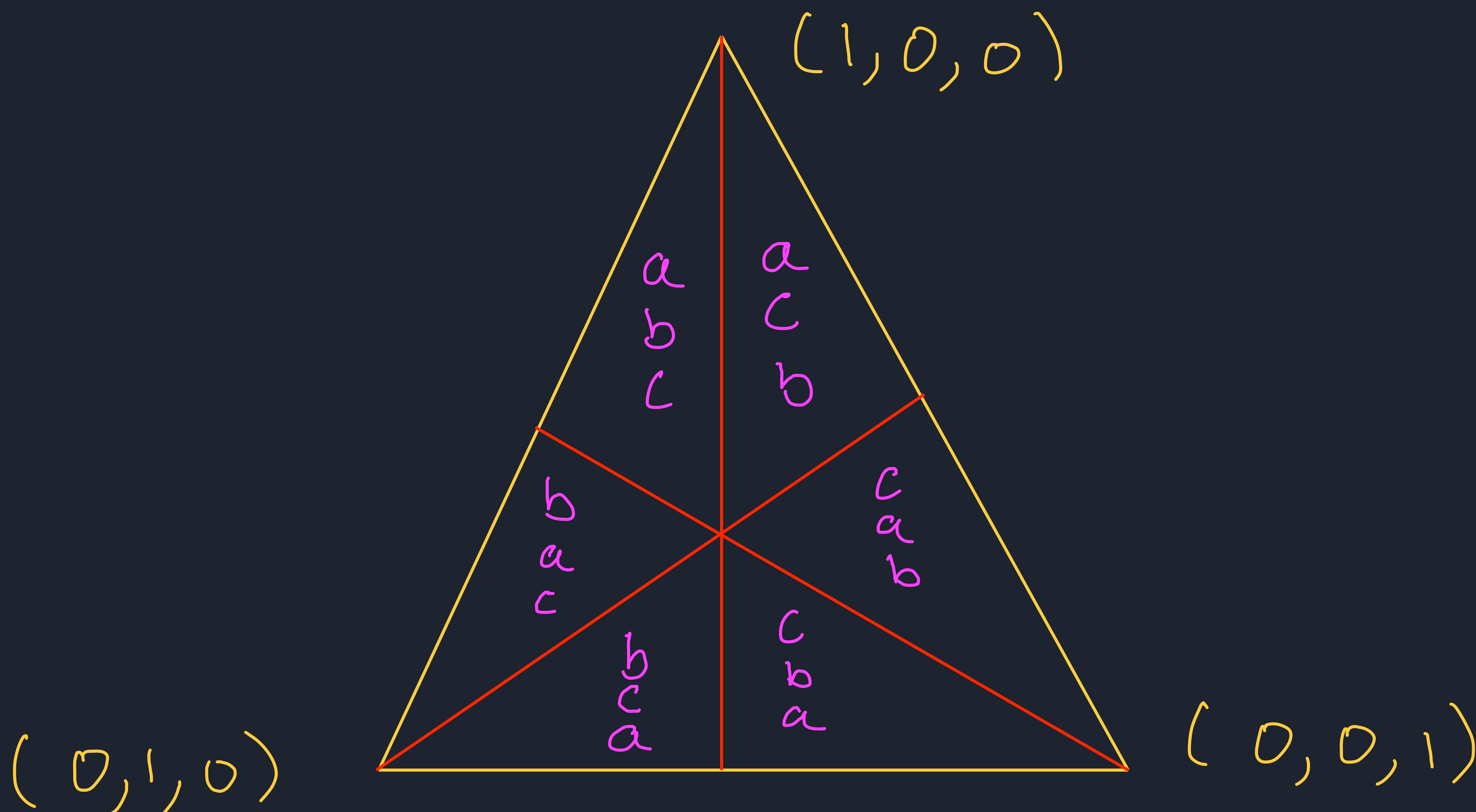


Partitioned by ordinal ranking of a, b & c

Why Ordinality?



Ordinality as an informational constraint:
Φ is measurable wrt this partition

Why Ordinality?

Theorem: $(|N| = |A| = 3)$

$$\varphi \text{ is } \left.\begin{array}{l} \text{efficient} \\ \text{strategy-proof} \\ \text{non-bossy} \\ \text{continuous} \end{array}\right\} \Longrightarrow \varphi \text{ is ordinal}$$

Why Ordinality?

Theorem: $(|N| = |A| = 3)$

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Ehlers, Majumdar, Mishra & Sen (2020) show a general result with a stronger continuity axiom

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Axioms for Ordinal Rules

Only need to bother with axioms that deal with comparing lotteries

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Non-bossiness & symmetry are the same

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Non-bossiness & symmetry are the same

The others replace EU comparisons with stochastic dominance comparisons

Axioms for Ordinal Rules

Stochastic dominance

Axioms for Ordinal Rules

Stochastic dominance

$$\pi_{ia_1} \geq \pi'_{ia_1}$$

best object according to u_i

Axioms for Ordinal Rules

Stochastic dominance

$$\pi_{ia_1} \geqslant \pi'_{ia_1}$$

$$\pi_{ia_2} + \pi_{ia_1} \geqslant \pi'_{ia_1} + \pi'_{ia_2}$$

Second best object according to u_i

Axioms for Ordinal Rules

Stochastic dominance

$$\pi_{ia_1} \geqslant \pi'_{ia_1}$$

$$\pi_{ia_2} + \pi_{ia_1} \geqslant \pi'_{ia_1} + \pi'_{ia_2}$$

$$\vdots$$

$$\pi_{ia_n} + \cdots + \pi_{ia_2} + \pi_{ia_1} \geqslant \pi'_{ia_1} + \pi'_{ia_2} + \cdots + \pi'_{ia_n}$$

Worst object according to u_{ia_n}

Axioms for Ordinal Rules

Stochastic dominance

$$\pi_{ia_1} \geq \pi'_{ia_1}$$

$$\pi_{ia_2} + \pi_{ia_1} \geq \pi'_{ia_1} + \pi'_{ia_2}$$

$$\vdots$$

$$\pi_{ia_n} + \dots + \pi_{ia_2} + \pi_{ia_1} \geq \pi'_{ia_1} + \pi'_{ia_2} + \dots + \pi'_{ia_n}$$

$$\pi_i \; u_i^{sd} \; \pi'_i$$

Axioms for Ordinal Rules

Stochastic dominance

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$$\pi_{ia_n} + \ldots + \pi_{ia_2} + \pi_{ia_1} \geqslant \pi'_{ia_1} + \pi'_{ia_2} + \ldots + \pi'_{ia_n}$$

Only relies on ordinal content of u_i:

$\forall\ u_i, u'_i \in \mathcal{U}^P \qquad u_i^{sd} = u_i'^{sd}$

Axioms for Ordinal Rules

Stochastic dominance

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$$\pi_{ia_n} + \dots + \pi_{ia_2} + \pi_{ia_1} \geqslant \pi'_{ia_1} + \pi'_{ia_2} + \dots + \pi'_{ia_n}$$

$\mathcal{U}_i^{sd}$ is an incomplete ordering of lotteries

Axioms for Ordinal Rules

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Comparison by expected utility according to u_i is one completion of u_i^{sd}

Axioms for Ordinal Rules

Substantial literature following B&M (2001) replaces EU comparison with sd comparison in other axioms

sd-Fairness

B&M (2001)

$\forall u \in \mathcal{U}^N, \ \forall i, j \in N$

$$\phi_i(u) \quad \mathcal{U}_i^{sd} \quad \phi_j(u)$$

or

$$\phi_i(u) = \phi_j(u)$$

sd-Fairness

Proposition: For every ordinal rule φ,

$$\varphi \text{ is fair} \iff \varphi \text{ is sd-fair}$$

sd–Strategy-proofness

B & M (2001)

$\forall u \in \mathcal{U}^N, \ \forall i \in N, \ \forall u_i' \in \mathcal{U}$

$$\varphi_i(u) \ u_i^{sd} \ \varphi_i(u_i', u_{-i})$$

or

$$\varphi_i(u) = \varphi_i(u_i', u_{-i})$$

sd–Strategy-proofness

Proposition: For every ordinal rule φ,

φ is strategy-proof $\iff$ φ is sd-strategy-proof

sd-Efficiency   B&M (2001)

$\forall u, \nexists \pi \in \Pi$ such that

$\forall i \quad \pi_i \; u_i^{sd} \; \phi_i(u)$

or

$\pi_i = \phi_i(u)$

$\exists i \quad \pi_i \; u_i^{sd} \; \phi_i(u)$

sd–Efficiency

Proposition: For every ordinal rule φ,

φ is efficient $\Longleftrightarrow$ φ is sd-efficient

sd–Efficiency

~~Proposition: For every ordinal rule φ,~~

~~φ is efficient $\iff$ φ is sd-efficient~~

A Consistent Definition of Efficiency

Incompleteness of u_i^{sd} is what makes things tricky

A Consistent Definition of Efficiency

Incompleteness of u_i^{sd} is what makes things tricky

Say that ϕ is sd-efficient[†] if

$\forall u \; \forall \pi \in \Pi$

$\forall i \qquad \phi_i(u) \quad u_i^{sd} \quad \pi_i$

or $\quad \phi_i(u) \quad = \quad \pi_i$

A Consistent Definition of Efficiency

Incompleteness of u_i^{sd} is what makes things tricky

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$$\text{or} \qquad \phi_i(u) \quad = \quad \pi_i$$

This is Gibbard's (1977) definition of ex ante efficiency

A Consistent Definition of Efficiency

Proposition: For every ordinal rule φ^s,

φ is efficient $\iff$ φ is sd-efficient$^+$

A Consistent Definition of Efficiency

Proposition: For every ordinal rule φ,[s]

φ is efficient $\Longleftrightarrow$ φ is sd-efficient[+]

Since we'll be working with ordinal rules, we'll drop the "sd-" prefixes and this +

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Efficiency vs sd-Efficiency

"No change makes everyone better off"

Efficiency vs sd-Efficiency

"No change makes everyone better off"

"Any change makes someone worse off"

Efficiency vs sd-Efficiency

"No change makes everyone better off"

These are essentially the same for complete preferences

"Any change makes someone worse off"

Efficiency vs sd-Efficiency

"No change makes everyone better off"

"Any change makes someone worse off"

This one is stronger for incomplete comparisons

Efficiency vs sd-Efficiency

sd-efficiency

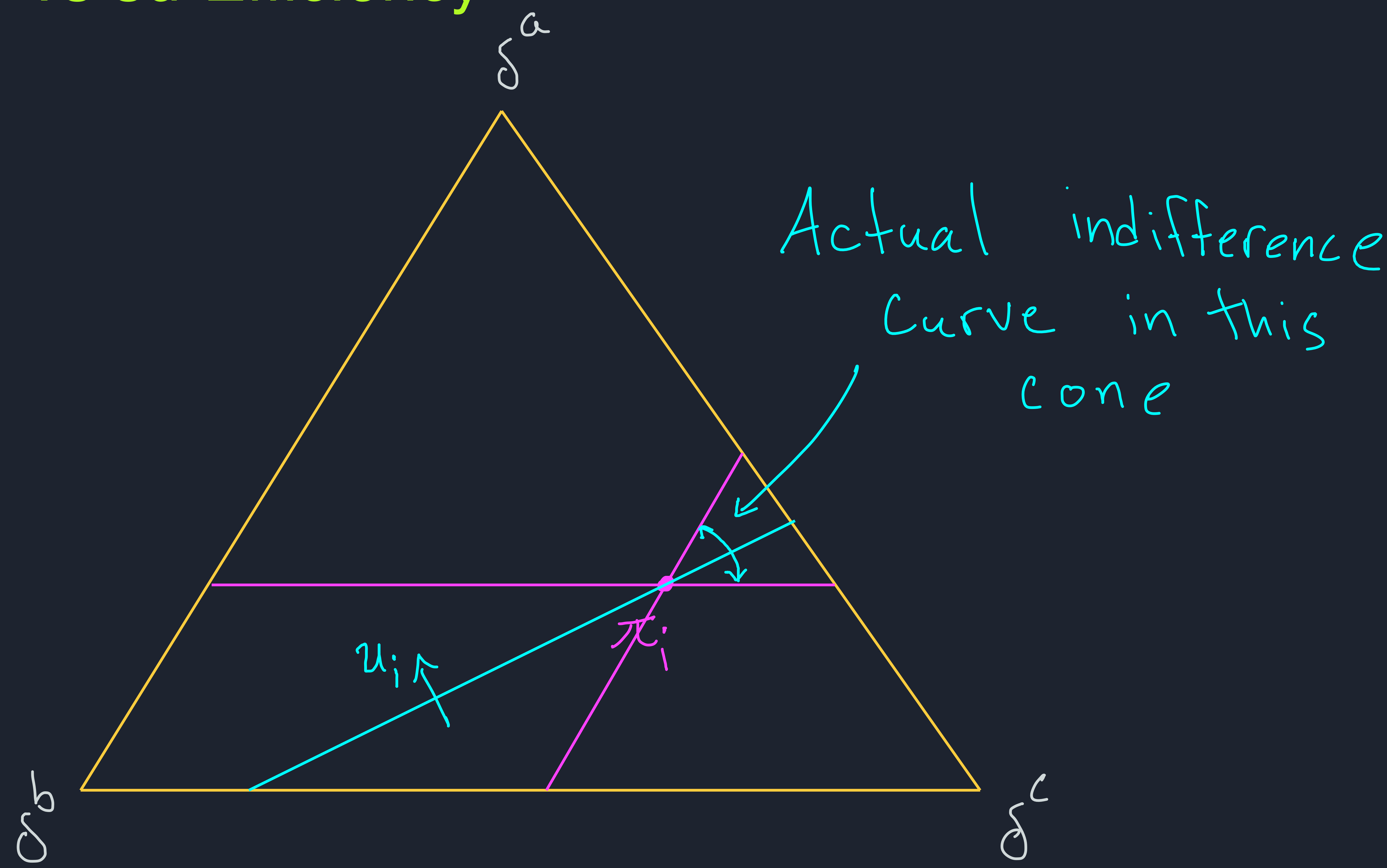
"No change makes everyone better off"

"Any change makes someone worse off"

our efficiency

Efficiency vs sd-Efficiency

$u_{ia} > u_{ib} > u_{ic}$



Efficiency vs sd-Efficiency

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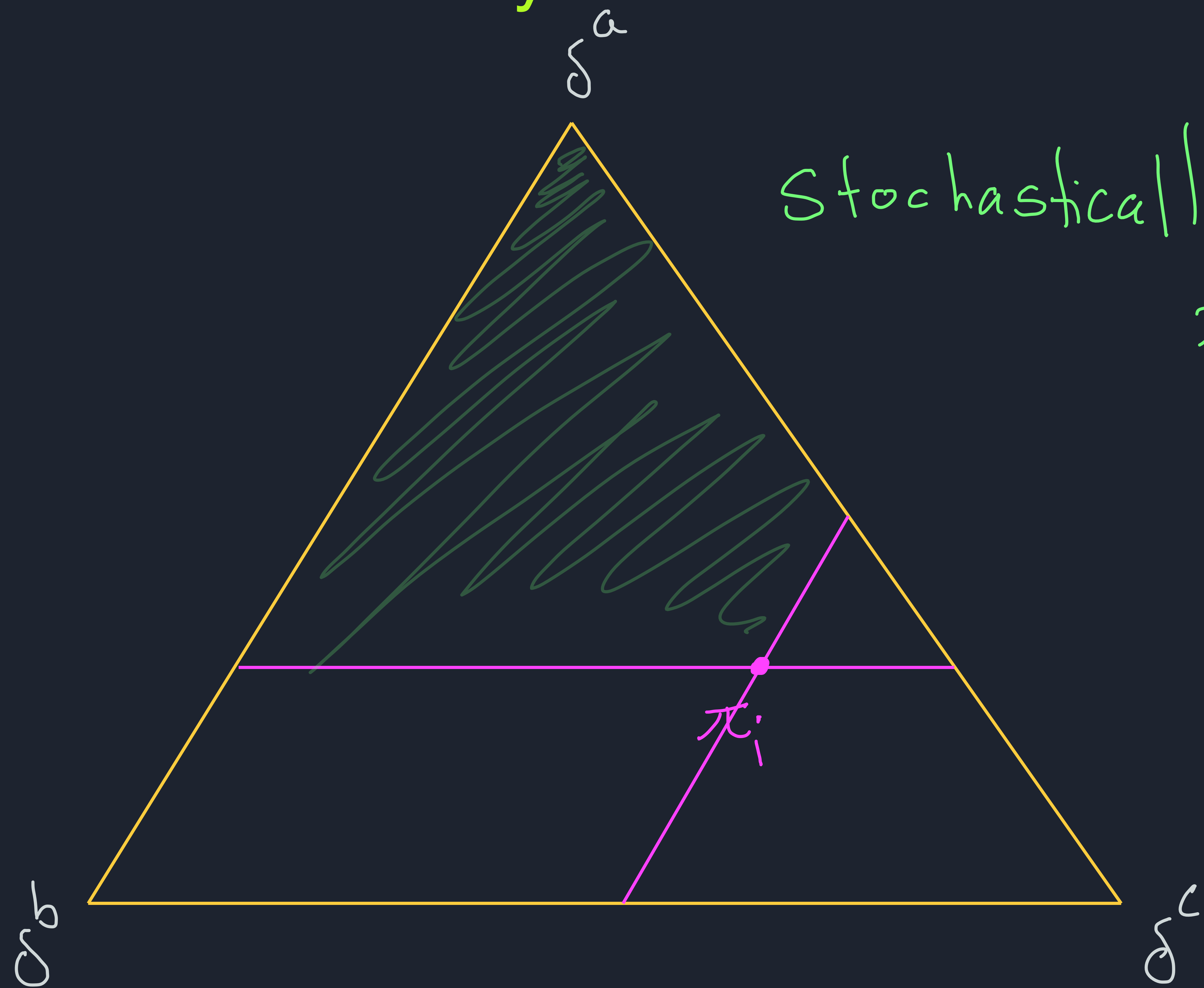

Stochastically dominates π_i

Efficiency vs sd-Efficiency

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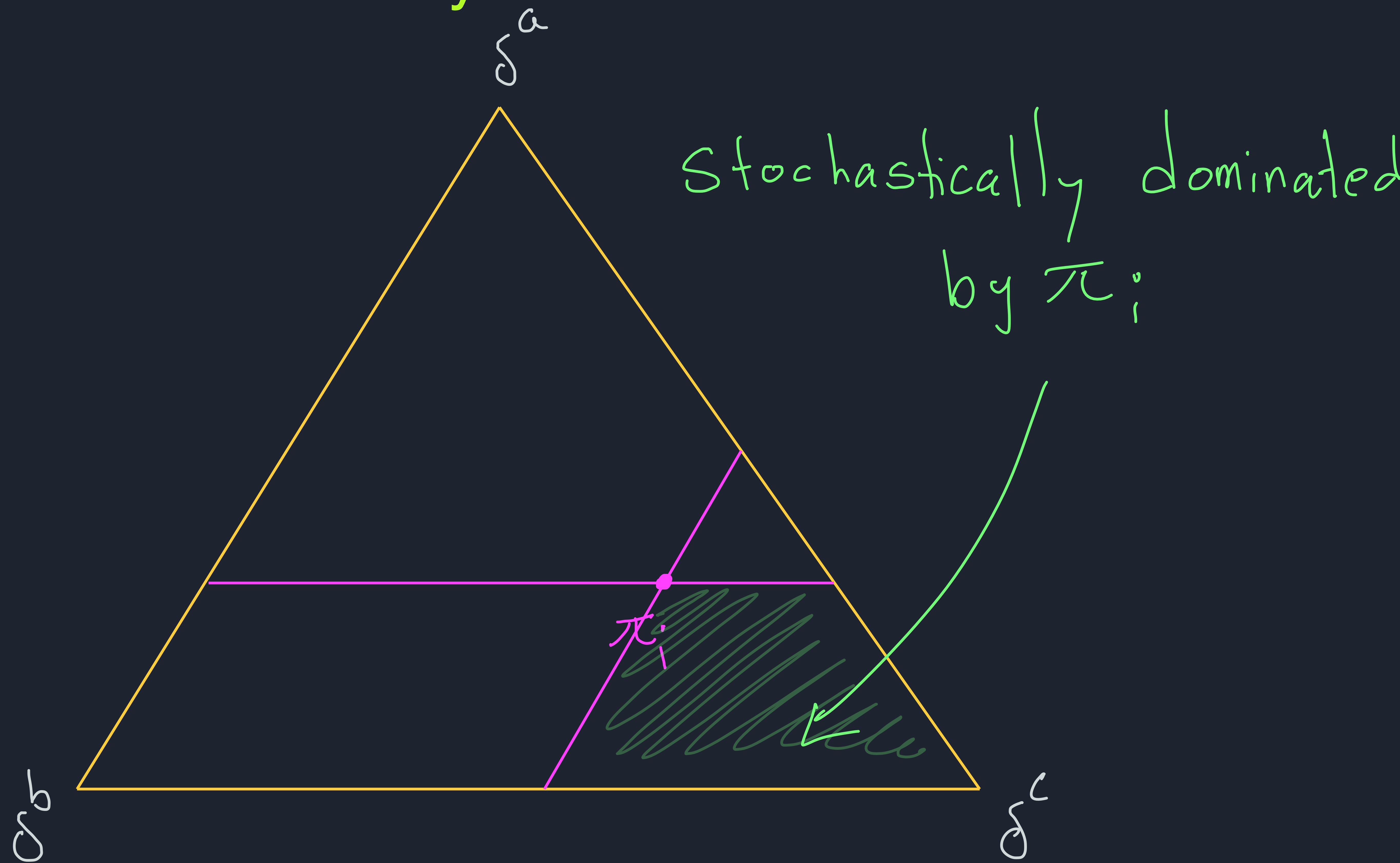

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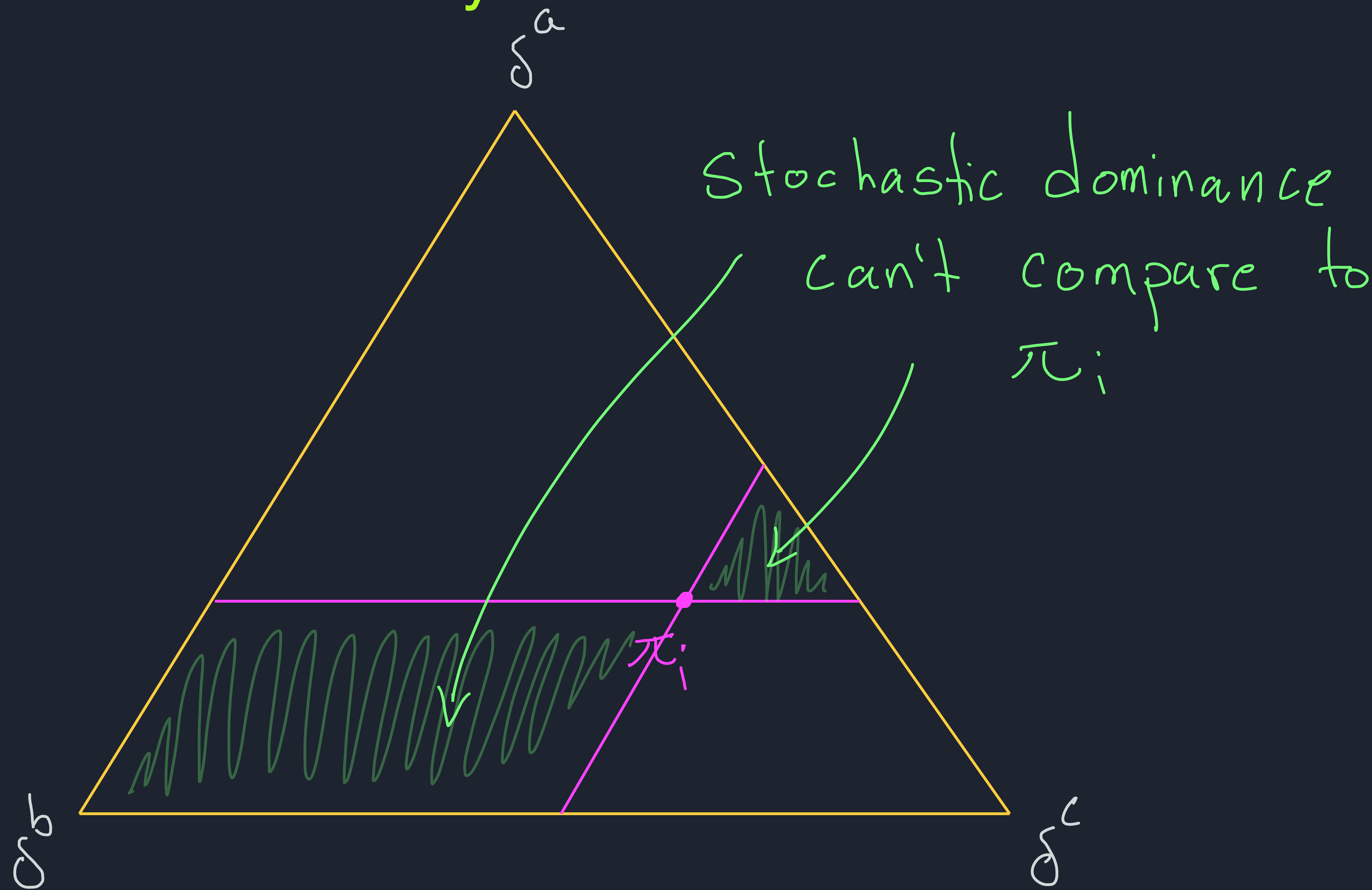
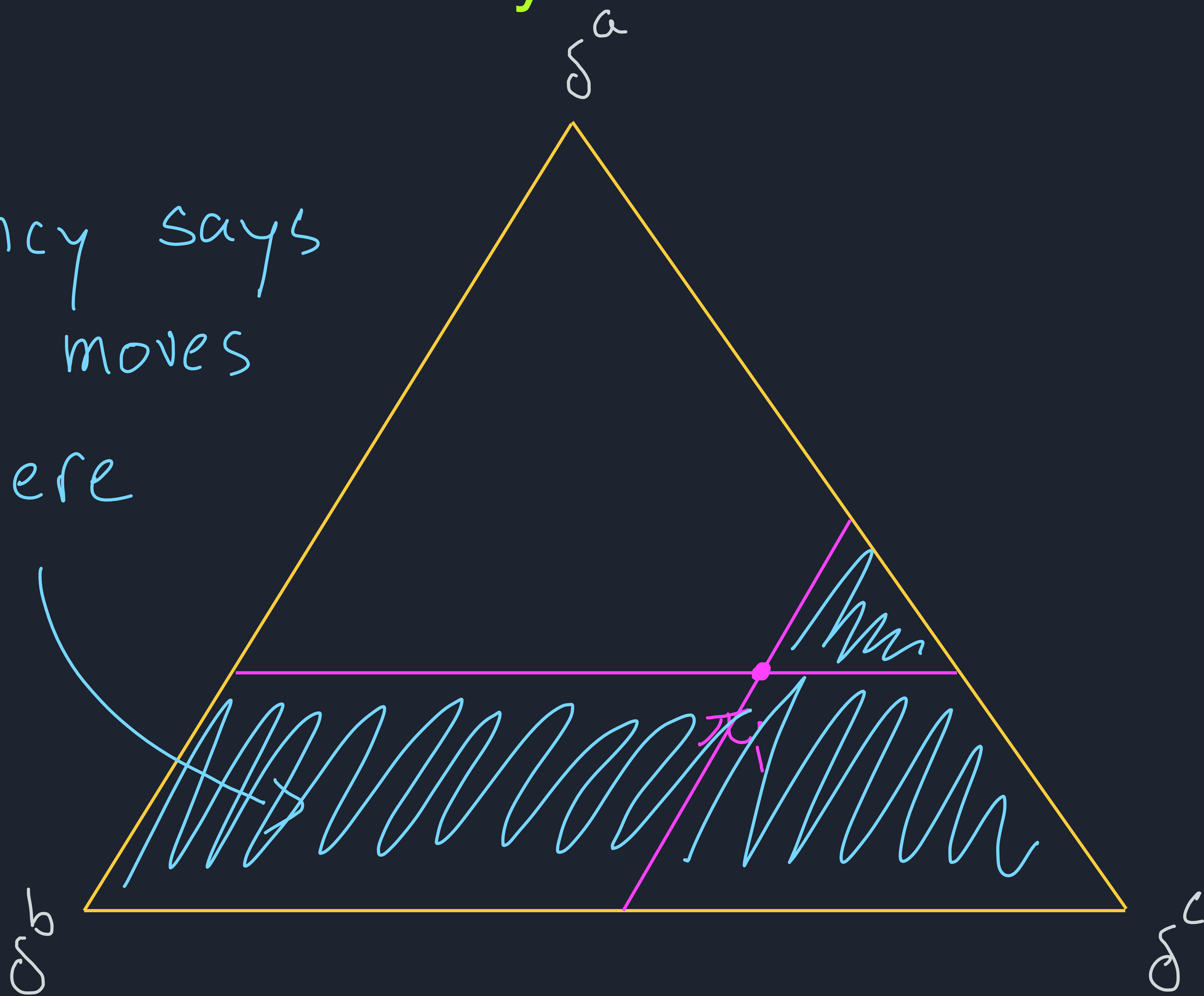


Efficiency vs sd-Efficiency

$u_{ia} > u_{ib} > u_{ic}$

sd-efficiency says any change moves someone here



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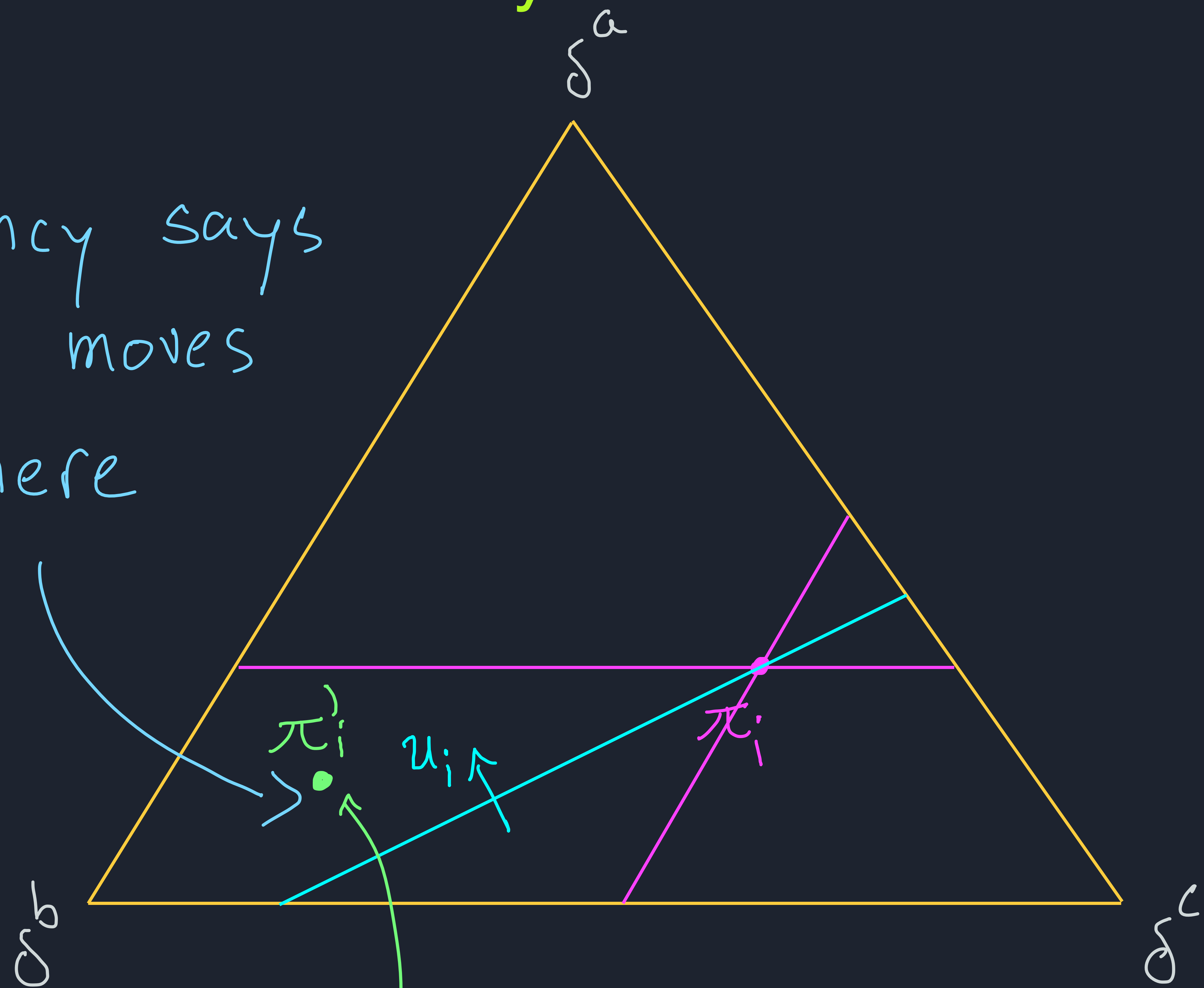

But $u_i \cdot \pi_i' > u_i \pi_i$!

Efficiency vs sd-Efficiency

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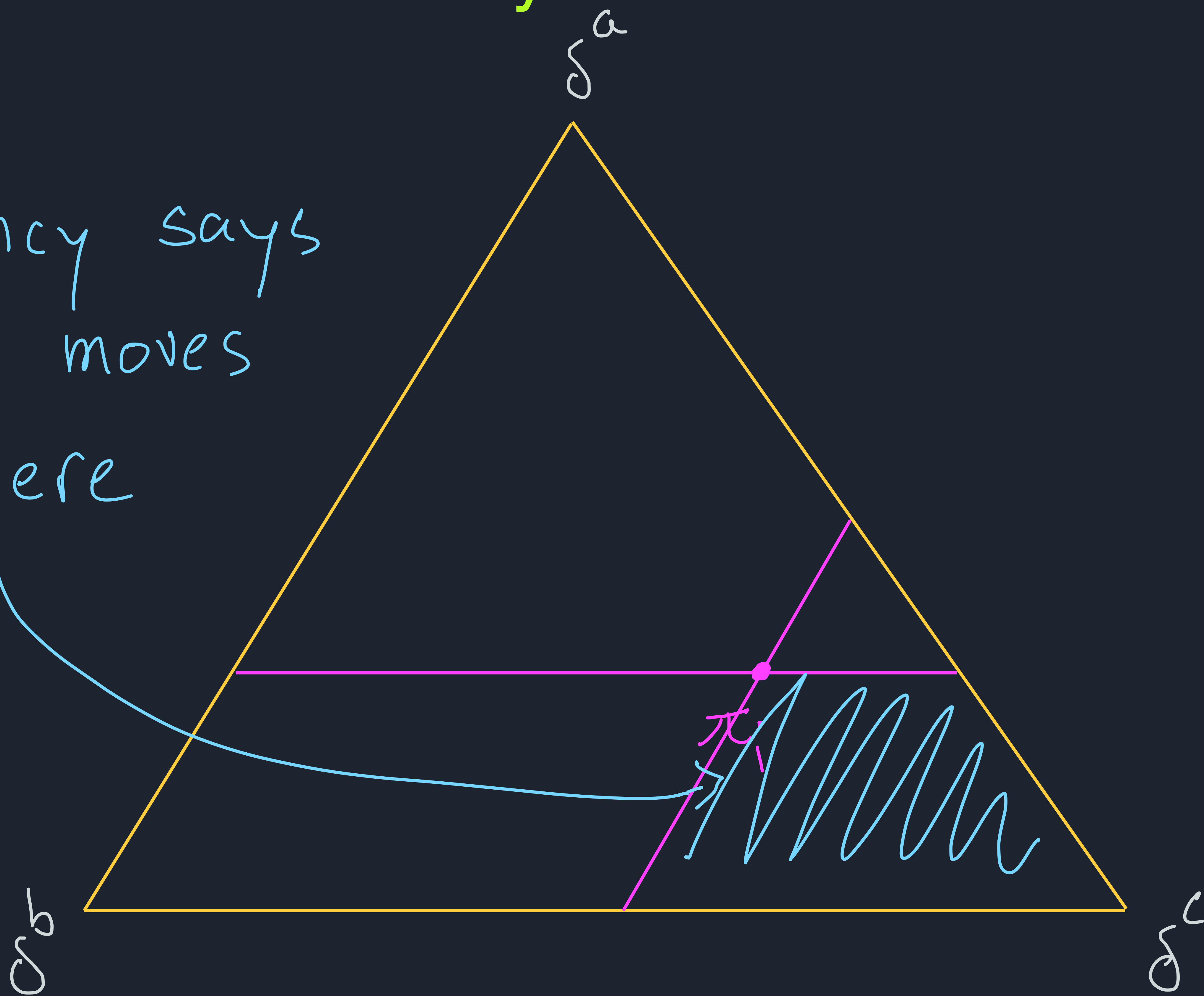
efficiency says any change moves someone here



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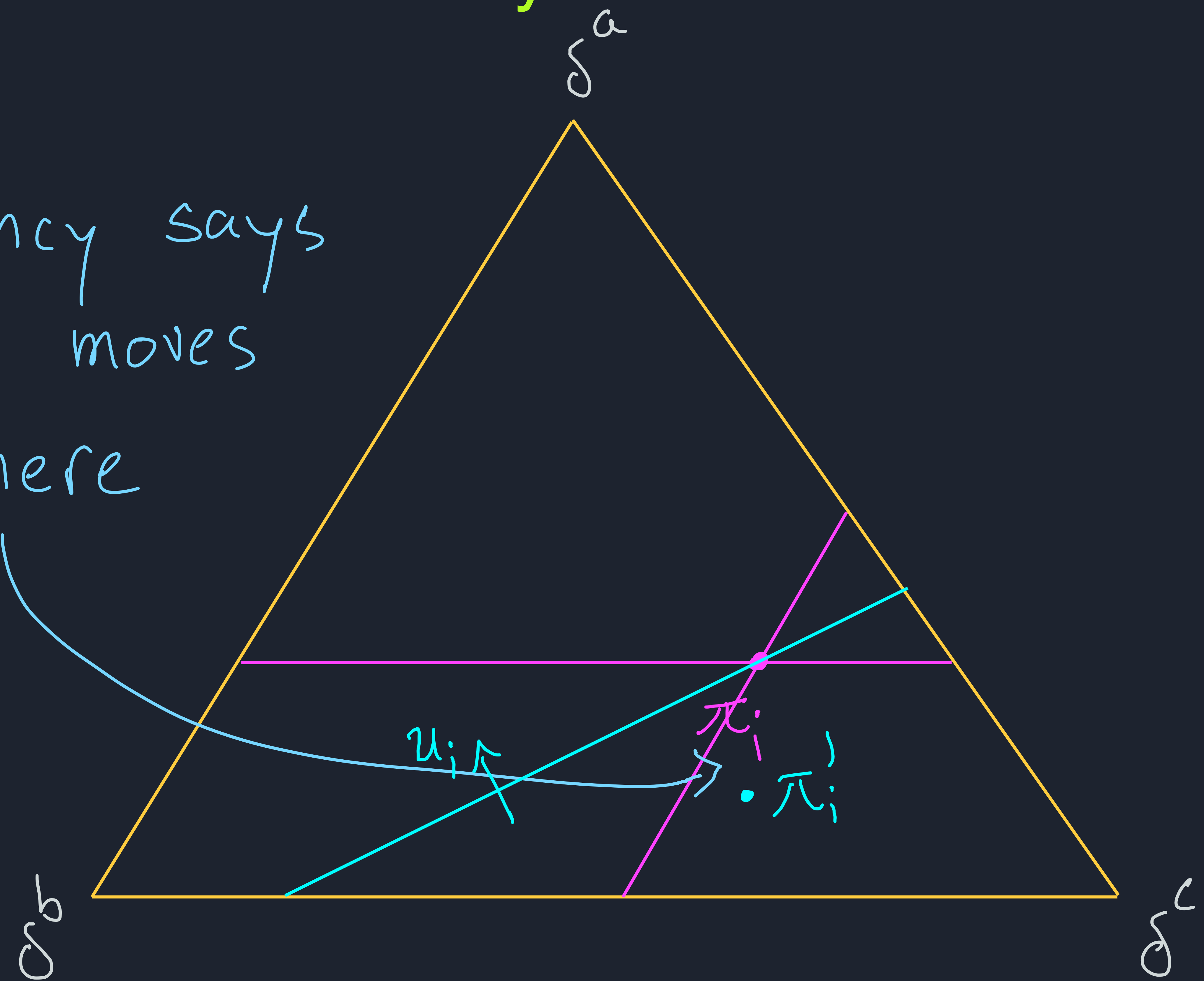


$u_i \pi_i > u_i \pi_i'$ as intended

Efficiency vs sd-Efficiency

B&M (2001) introduce sd-efficiency and define a new rule that is sd-fair and sd-efficient

Efficiency vs sd-Efficiency

B&M (2001) introduce sd-efficiency and define a new rule that is fair and sd-efficient

Efficiency vs sd-Efficiency

B & M (2001) introduce sd-efficiency and define a new rule that is fair and sd-efficient

Fair $\Rightarrow$ symmetric

Efficiency vs sd-Efficiency

B & M (2001) introduce sd-efficiency and define a new rule that is fair and sd-efficient

Fair $\Rightarrow$ symmetric

Given the weakness of sd-efficiency how inefficient could a symmetric rule be?

Cost of Symmetry

Measure of worst case welfare loss for an agent i at π:

$$\max_{\pi' \text{ that dominates } \pi} \frac{u_i \cdot \pi'_i - u_i \cdot \pi_i}{\max_a u_{ia}}$$

skip

Cost of Symmetry

Measure of worst case welfare loss for an agent i at π:

$$\max_{\pi' \text{ that dominates } \pi} \underbrace{\frac{u_i \cdot \pi'_i - u_i \cdot \pi_i}{\max_a u_{ia}}}_{\text{Equivalent to gaining this much probability of best object to move from } \pi \text{ to } \pi'}$$

Cost of Symmetry

Measure of worst case welfare loss for an agent i at π:

$$\max_{\pi' \text{ that dominates } \pi} \frac{u_i \cdot \pi'_i - u_i \cdot \pi_i}{\max_a u_{ia}}$$

denote this $K_i(\pi)$

Cost of Symmetry

Measure of worst case welfare loss at π:

$$L(\pi) = \max_i K_i(\pi)$$

Cost of Symmetry

Measure of worst case welfare loss at π:

$$L(\pi) = \max_i K_i(\pi)$$

By definition, $\forall \pi \quad 0 \le L(\pi) \le 1$

Cost of Symmetry

Example with $|N| = |A| = 3$

	u_1	u_2	u_3
a	1	1	1
b	$1-\varepsilon$	ε	ε
c	0	0	0

Cost of Symmetry

Example with $|N| = |A| = 3$

	u_1	u_2	u_3
a	1	1	1
b	$1-\varepsilon$	ε	ε
c	0	0	0

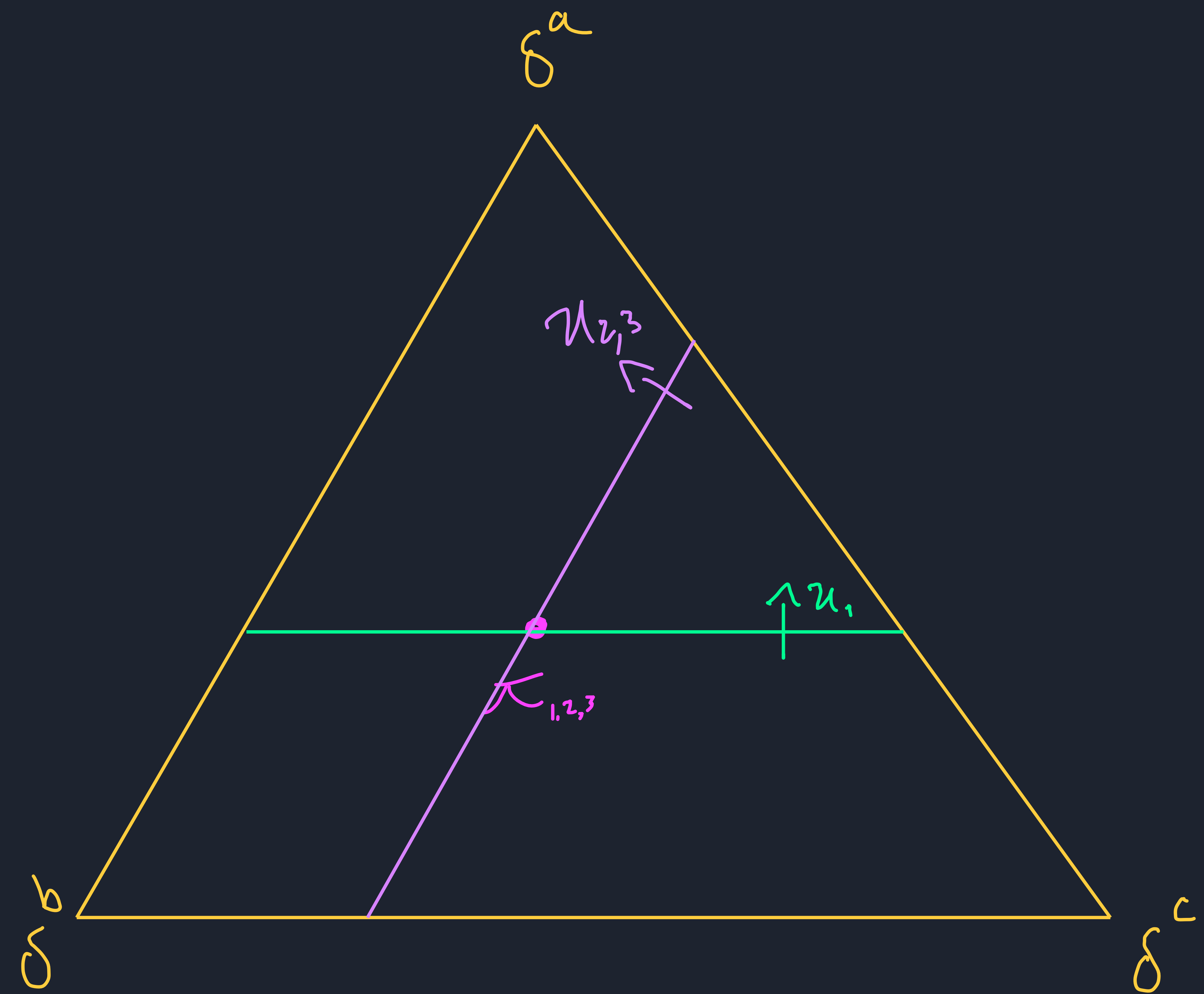
ϕ Ordinal symmetric

$\Downarrow$

$\forall i \quad \phi_i(u) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

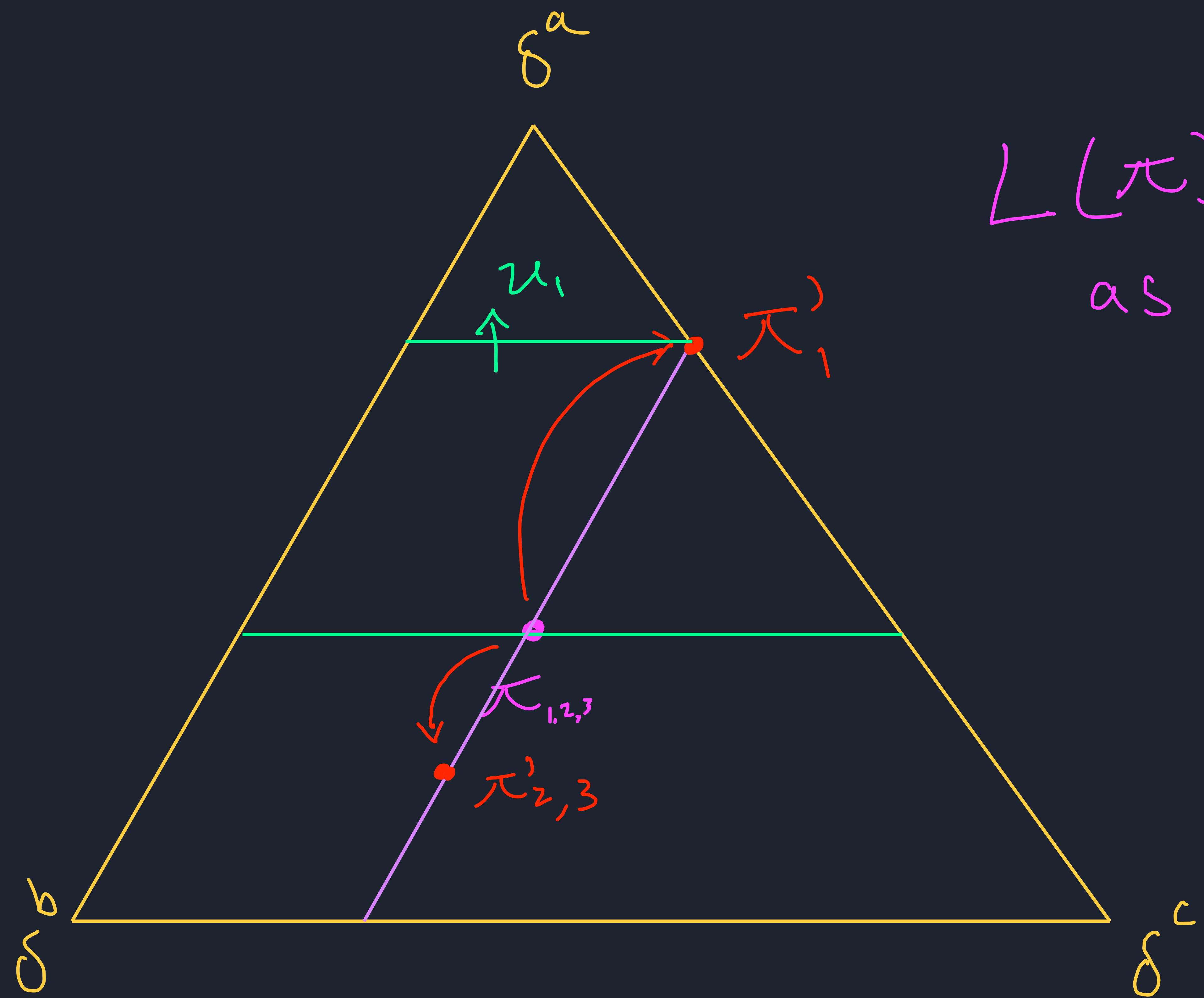
Cost of Symmetry

Example with $|N| = |A| = 3$



Cost of Symmetry

Example with $|N| = |A| = 3$



$L(\pi) \longrightarrow \frac{1}{3}$

as $\varepsilon \longrightarrow 0$

Cost of Symmetry

Extending this example to $|N| = |A| = n$

$$L(\pi) \longrightarrow \frac{n-2}{n}$$

So $L(\pi)$ can get arbitrarily close to 1 for n large enough and ε small enough

Cost of Symmetry

Extending this example to $|N| = |A| = n$

$$L(\pi) \longrightarrow \frac{n-2}{n}$$

For ordinal rules, symmetry can come at a substantial loss of efficiency

A Little Background

A Little Background

Most obvious rule:

Random Serial Dictatorship

A Little Background

Most obvious rule:

Random Serial Dictatorship

Ubiquitous

A Little Background

Most obvious rule:

Random Serial Dictatorship

Ex post efficient

A Little Background

Most obvious rule:

Random Serial Dictatorship

Not efficient in our (ex ante) sense

A Little Background

Most obvious rule:

Random Serial Dictatorship

Not even sd-efficient

A Little Background

Most obvious rule:

Random Serial Dictatorship

Not even sd-efficient

Hylland & Zeckhauser (1979) suggest Competitive Equilibrium from Equal Income as an efficient alternative

A Little Background

CEEI $\longleftarrow$ not strategy-proof

A Little Background

CEEI ← not strategy-proof

But symmetric and efficient

A Little Background

CEEI $\longleftarrow$ not strategy-proof

But symmetric and efficient

Zhou (1990):

$\nexists \ \phi$ efficient
strategy-proof
symmetric

A Little Background

CEEI $\leftarrow$ not strategy-proof

But symmetric and efficient

Zhou (1990):

$\nexists\ \phi$ efficient, strategy-proof, symmetric $\Big\}$ Fundamental tension between these three

A Little Background

Obviously, Zhou's (1990) result means

$\nexists \ \varphi$ efficient
strategy-proof
symmetric
ordinal

A Little Background

B&M (2001):

$\nexists\ \phi$ sd-efficient
strategy-proof
symmetric
ordinal

A Little Background

As we just saw:

$\nexists\ \phi$ efficient

symmetric

ordinal

A Little Background

B&M (2001):

$\exists\, \phi$ sd-efficient

Symmetric

Ordinal

They define the "Probabilistic Serial" rule that has these properties

A Little Background

Large literature on PS

A Little Background

Large literature on PS

PS trades strategy-proofness of RSD
for an efficiency gain

A Little Background

Large literature on PS

PS trades strategy-proofness of RSD for an efficiency gain

But that efficiency gain might not be what one hopes for

Efficiency vs sd-Efficiency

If efficiency is key, maybe sd-efficiency isn't the right formulation

Efficiency vs sd-Efficiency

If efficiency is key, maybe sd-efficiency isn't the right formulation

Instead, our notion of efficiency might be the one to think of

Efficiency vs sd-Efficiency

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As we'll see, there is a rich class of strategy-proof and efficient rules

Efficiency vs sd-Efficiency

If efficiency is key, maybe sd-efficiency isn't the right formulation

Instead, our notion of efficiency might be the one to think of

As we'll see, there is a rich class of strategy-proof and efficient rules (though they aren't symmetric)

Efficiency vs sd-Efficiency

One more way to compare

Efficiency vs sd-Efficiency

One more way to compare

RSD uniformly randomizes over orderings

Efficiency vs sd-Efficiency

One more way to compare

RSD uniformly randomizes over orderings

Can limit randomization

Efficiency vs sd-Efficiency

One more way to compare

RSD uniformly randomizes over orderings

Can limit randomization

Harless & Phan (2022) characterize maximal sets of orders that you can randomize over and still get sd-efficiency

Efficiency vs sd-Efficiency

Obviously singleton sets give you efficiency

Efficiency vs sd-Efficiency

Obviously singleton sets give you efficiency

Can do more

Efficiency vs sd-Efficiency

Obviously singleton sets give you efficiency

Can do more

B&M (2001) already showed:

RSD is sd-efficient for $|N|=3$

Efficiency vs sd-Efficiency

Obviously singleton sets give you efficiency

Can do more

B&M (2001) already showed:
RSD is sd-efficient for $|N|=3$

So randomizing uniformly over three agents should work

Efficiency vs sd-Efficiency

$\mathcal{O}$ - a set of orders

Efficiency vs sd-Efficiency

Θ - a set of orders

$\succ \in \Theta \qquad \succ' \in \Theta$

Efficiency vs sd-Efficiency

Θ - a set of orders

$\succ$		$\succ'$
⋮	exactly the same	⋮
i	can be anything	k
j		i
k		j
⋮	exactly the same	⋮

Efficiency vs sd-Efficiency

Θ - a set of orders

$\succ$ | $\succ'$

exactly the same

i | k

j | i

k | j

can be anything

exactly the same

H&P (2022) call these "adjacent-three" sets of orders

Efficiency vs sd-Efficiency

Adjacent-three sets are maximal to ensure RSD is sd-efficient

Efficiency vs sd-Efficiency

Adjacent-three sets are maximal to ensure RSD is sd-efficient

What about efficiency?

Efficiency vs sd-Efficiency

Adjacent-three sets are maximal to ensure RSD is sd-efficient

What about efficiency?

RSD for $|N|=3$ is symmetric

Efficiency vs sd-Efficiency

Adjacent-three sets are maximal to ensure RSD is sd-efficient

What about efficiency?

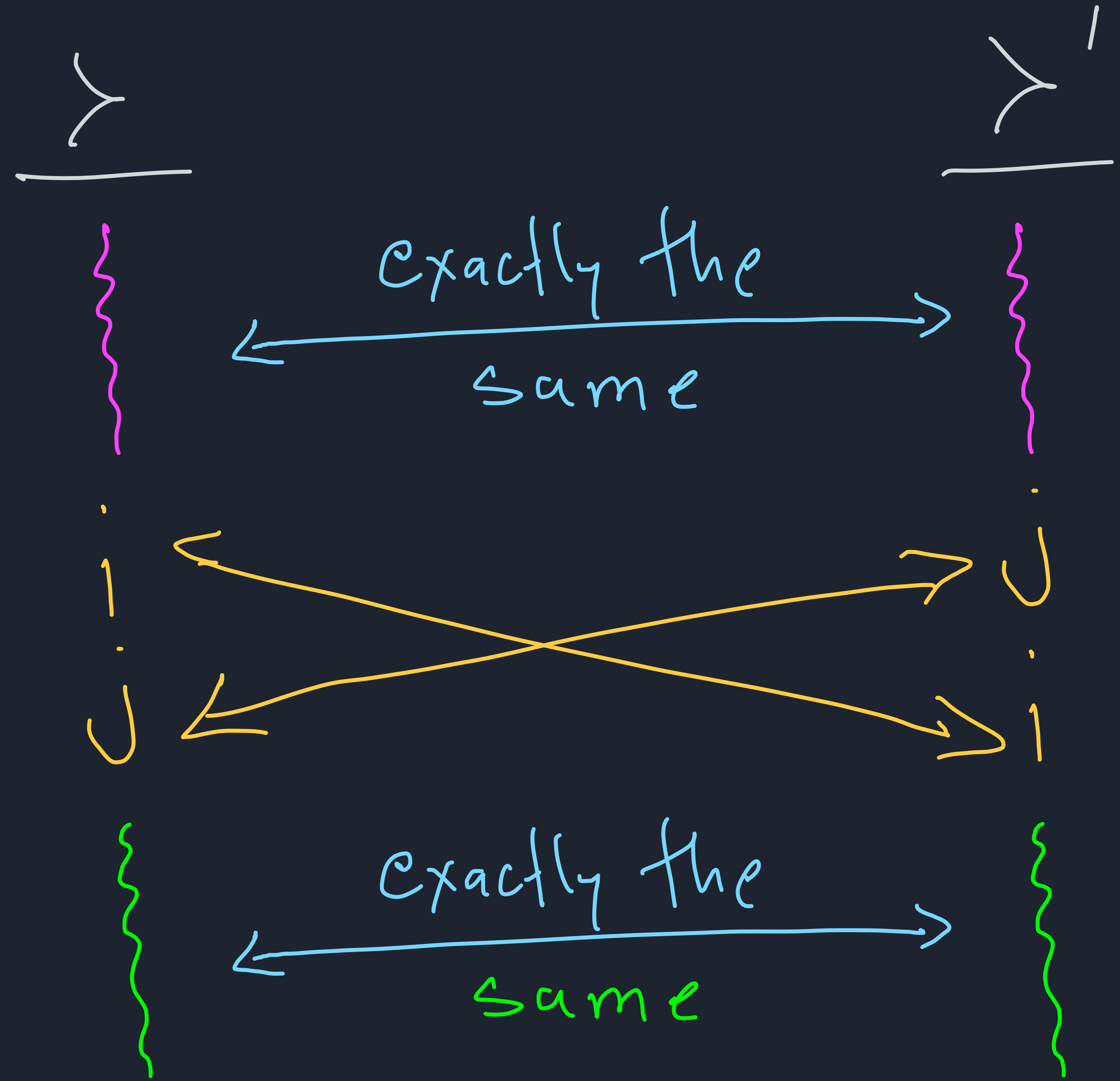
RSD for $|N|=3$ is symmetric

$\Downarrow$

As we saw not efficient

Efficiency vs sd-Efficiency

Θ - a set of orders



"Adjacent-two"

Efficiency vs sd-Efficiency

Proposition: Adjacent-two sets are maximal for RSD to be efficient

Efficiency vs sd-Efficiency

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Lots of reasons to use RSD

Efficiency vs sd-Efficiency

Proposition: Adjacent-two sets are maximal for RSD to be efficient

Lots of reasons to use RSD

Simplicity (Li (2017); Pycia & Troyan (2023))
is the main one

Efficiency vs sd-Efficiency

Proposition: Adjacent-two sets are maximal for RSD to be efficient

Lots of reasons to use RSD

Simplicity (Li (2017); Pycia & Troyan (2023)) is the main one

Then cost of efficiency over sd-efficiency is going from adjacent-3 to adjacent-2

Outline of the Talk

- ✓ The random allocation model
- ✓ An argument for ordinal allocation rules
- ✓ Porting axioms from the cardinal model to the ordinal model
- ✓ Contrast our efficiency to the rest of the literature
- ☆ A characterization of ex ante efficient, strategy-proof, ... rules

A Characterization

φ is RSD over adjacent-two $\Rightarrow$ φ is efficient, strategy-proof, non-bossy, continuous

A Characterization

φ is RSD over adjacent-two $\Rightarrow$ $\not\Leftarrow$ φ is efficient, strategy-proof, non-bossy, continuous

A Characterization

Need to add an axiom

A Characterization

Need to add an axiom

φ is neutral if renaming objects
is inconsequential

May (1952)

A Characterization

Proposition: for $|N| = 3$

φ is RSD over adjacent-two $\iff$ φ is efficient, strategy-proof, non-bossy, ordinal, neutral

A Characterization

Proposition: for $|N| = 3$

φ is RSD over adjacent-two $\iff$ φ is efficient
strategy-proof
non-bossy
continuous
neutral

From earlier theorem

A Characterization

Proposition: for $|N| = 3$

φ is RSD over adjacent-two $\iff$ φ is efficient, strategy-proof, non-bossy, ordinal, neutral

Parallel with Gibbard's (1977) result for probabilistic Arrovian setting

Limited Randomization

Very limited randomization not just due to ordinality & efficiency (other axioms play a role)

Limited Randomization

Very limited randomization not just due to ordinality & efficiency (other axioms play a role)

Ordinality & efficiency $\Rightarrow \forall i, j$

$$|\mathrm{supp}(\pi_i) \cap \mathrm{supp}(\pi_j)| \leq 2$$

Limited Randomization

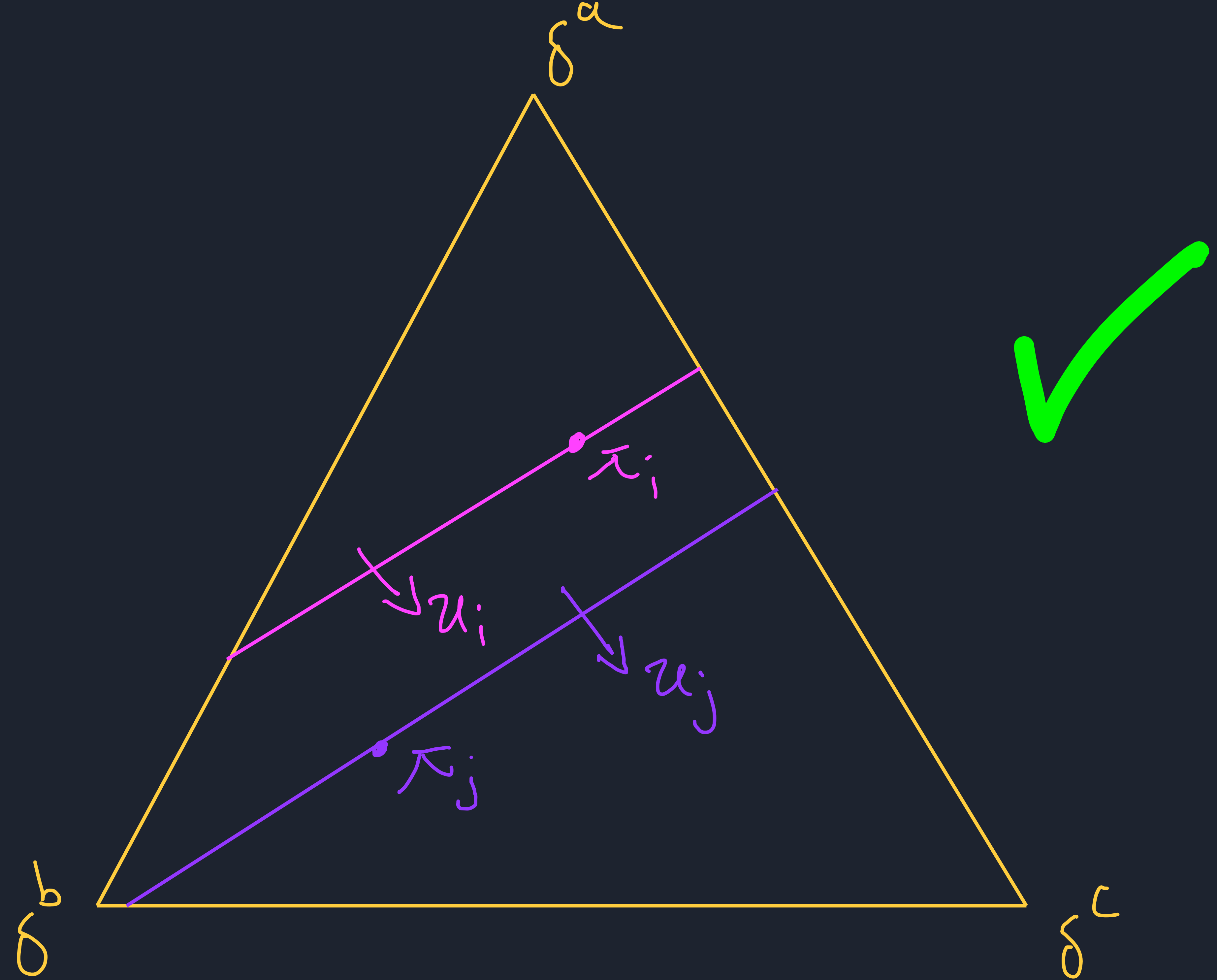
Very limited randomization not just due to ordinality & efficiency (other axioms play a role)

Ordinality & efficiency $\Rightarrow \forall i, j$

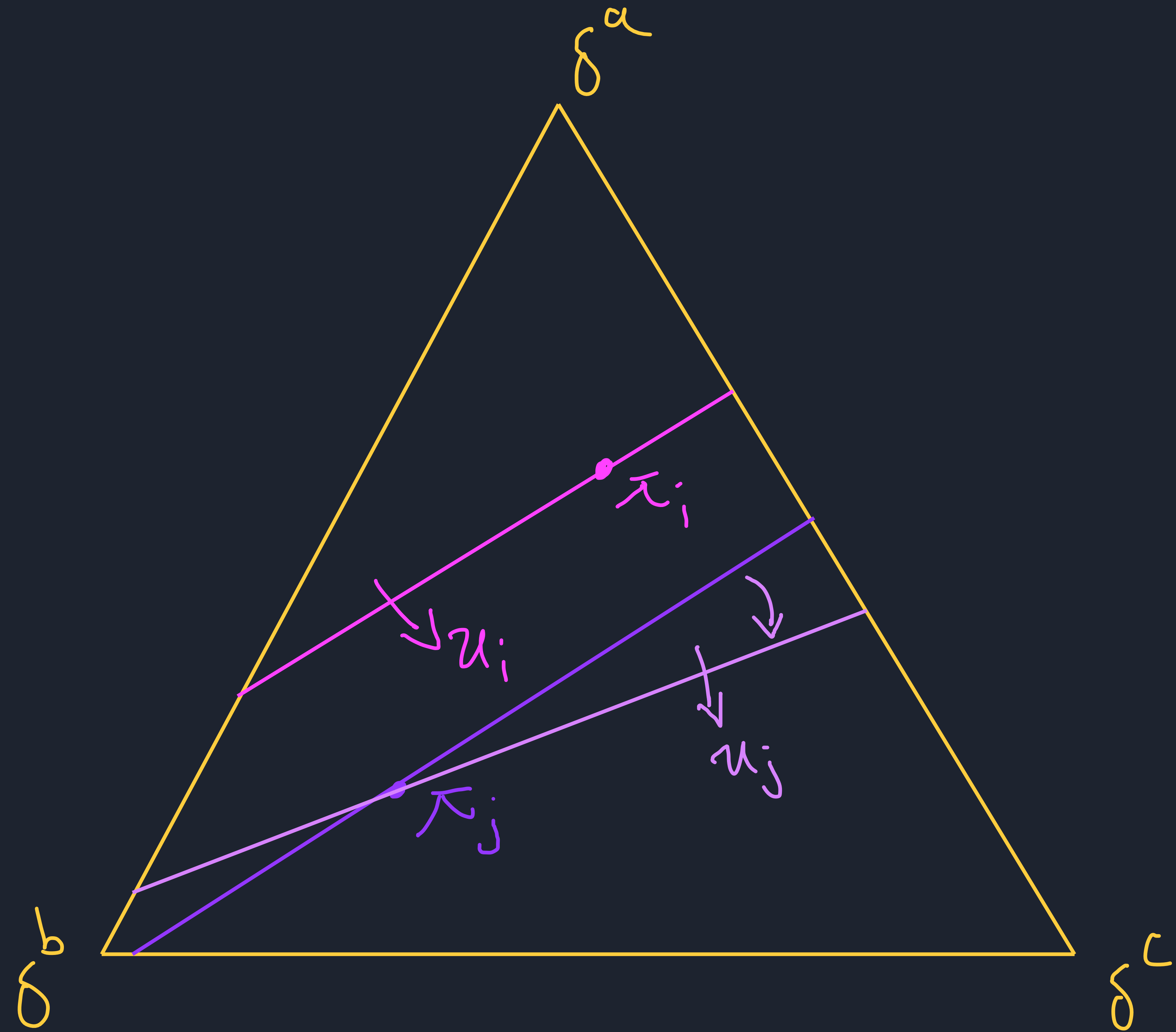
$$|\operatorname{supp}(\pi_i) \cap \operatorname{supp}(\pi_j)| \leq 2$$

This is generically the case even without ordinality

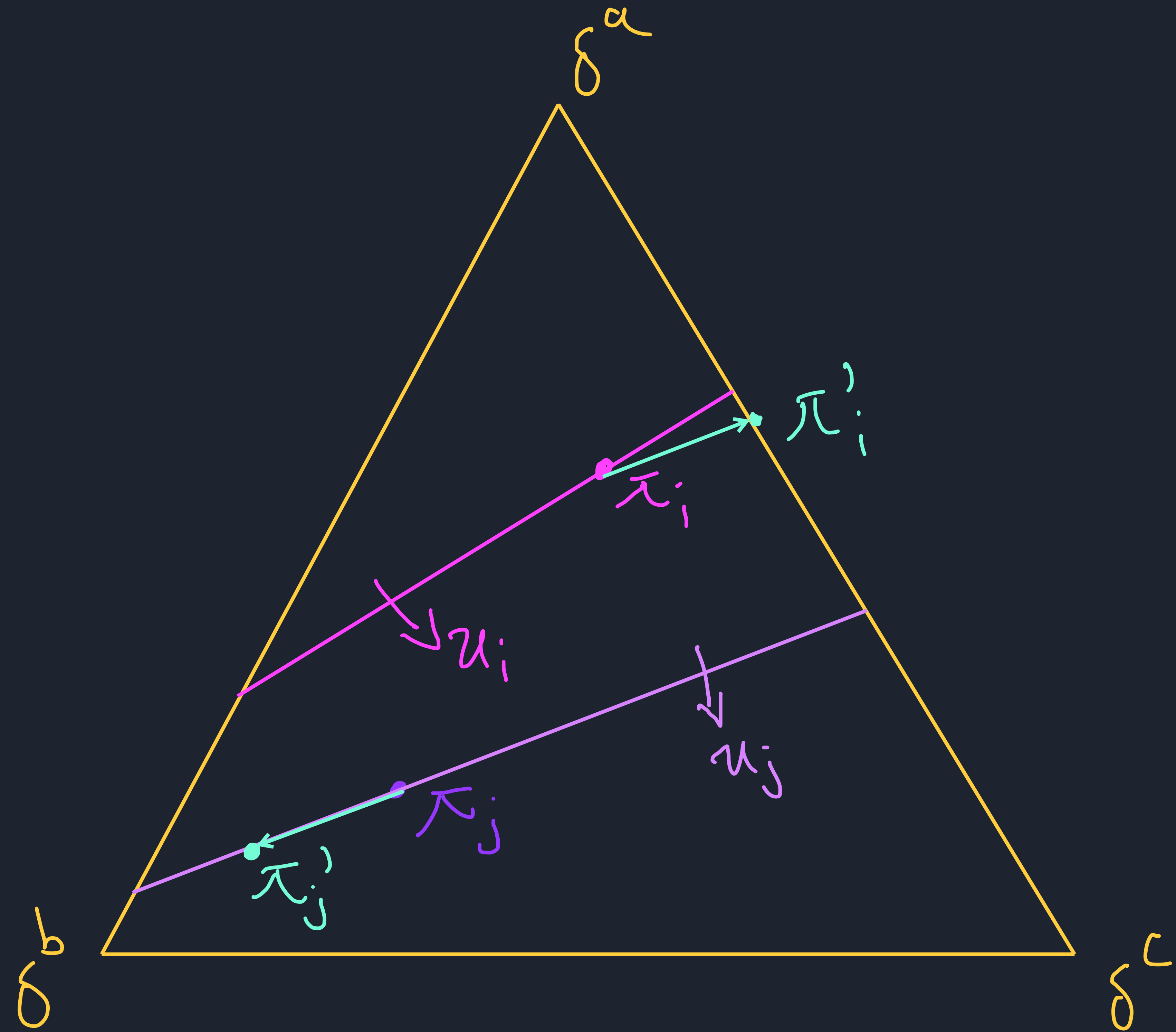
Limited Randomization



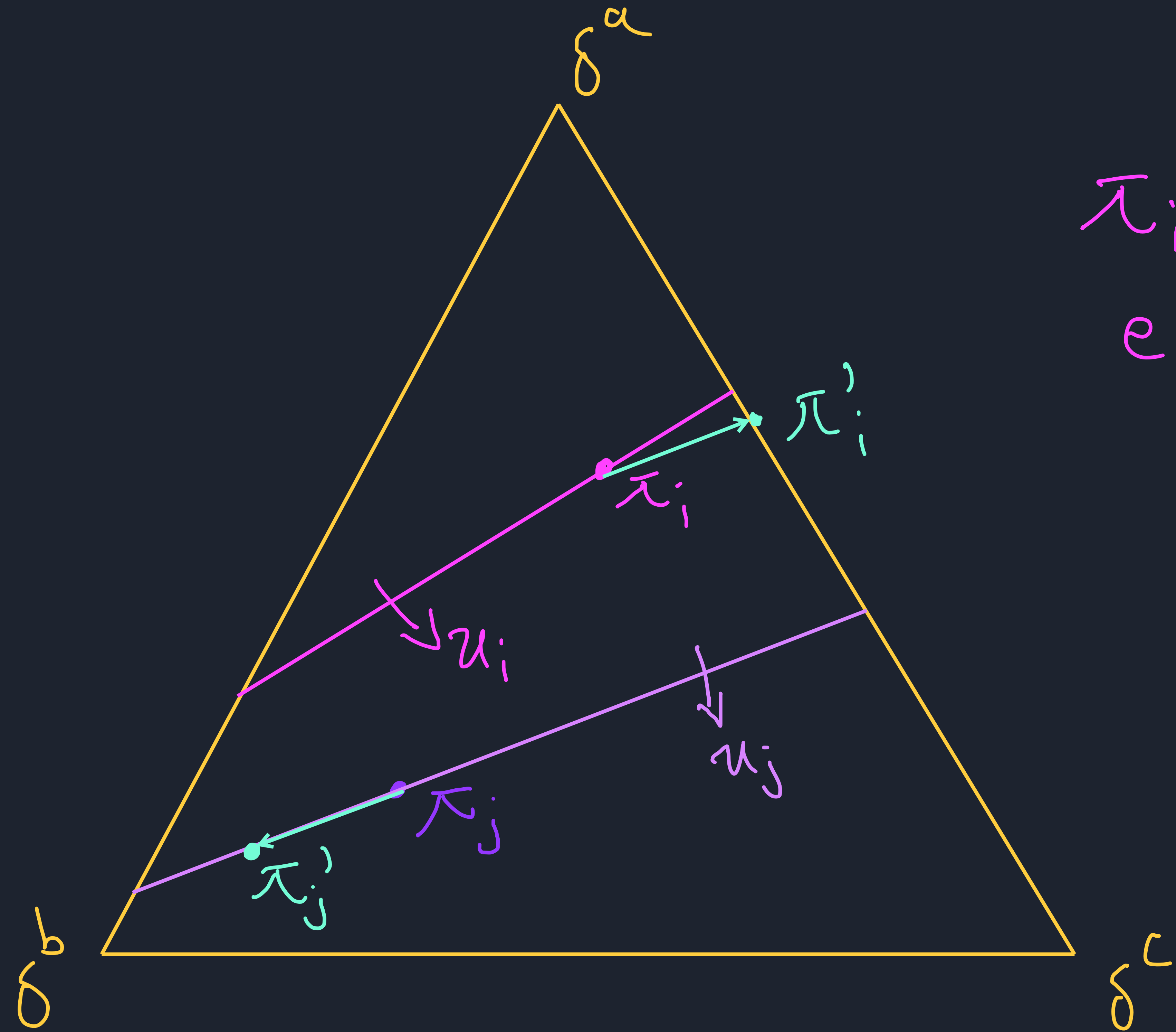
Limited Randomization



Limited Randomization



Limited Randomization



π_i not efficient

Back to the Characterization

for $|N| > 3$

φ is RSD over adjacent-two $\Rightarrow$ φ is efficient, strategy-proof, non-bossy, ordinal, neutral

A Characterization

for $|N| > 3$

φ is RSD over adjacent-two $\Longrightarrow$ φ is
- efficient
- strategy-proof
- non-bossy
- ordinal
- neutral

$\not\Longleftarrow$

Skip

A Class of Recursive Rules

Some notation to set us up

A Class of Recursive Rules

Some notation to set us up

$$\mathcal{S} = [0,1]^A \leftarrow \text{supply vectors}$$

A Class of Recursive Rules

Some notation to set us up

$\mathcal{S}$

$\overline{\Pi} \leftarrow$ Partial allocations

$\left(\forall i \quad \pi_i = 0 \text{ or } \sum_{a \in A} \pi_{ia} = 1 \right)$

A Class of Recursive Rules

Some notation to set us up

$\mathcal{S}$

$\overline{\Pi}$

$f : \mathcal{S} \times \mathcal{U}^N \longrightarrow \overline{\Pi} \longleftarrow$ partial allocation rule

$f(s, u) \in \overline{\Pi}$

$\forall a \quad \sum_i f_{ia}(s, u) \leq s_a$

A Class of Recursive Rules

Some notation to set us up

$\mathcal{S}$

$\overline{\overline{\Pi}}$

$f : \mathcal{S} \times \mathcal{U}^N \longrightarrow \overline{\overline{\Pi}}$ $\mathcal{F}$ - all partial rules

A Class of Recursive Rules

Some notation to set us up

$\mathcal{S}$

Π

$f : \mathcal{S} \times \mathcal{U}^N \longrightarrow \Pi \qquad \mathcal{F}$

$r(s, \pi) = \left(s_a - \sum_i \pi_{ia}\right)_{a \in A}$ ⟵ residual supply after allocating π from s

A Class of Recursive Rules

Some notation to set us up

$$\mathcal{S}$$

$$\overline{\Pi}$$

$$f : \mathcal{S} \times \mathcal{U}^N \longrightarrow \overline{\Pi} \qquad \mathcal{F}$$

$$r(s, \pi)$$

$$\eta = (\pi^1, \ldots, \pi^k) \qquad \eta \mathbin{++} \pi^{k+1} = (\pi^1, \ldots, \pi^k, \pi^{k+1})$$

append operator

A Class of Recursive Rules

$\mathcal{H} \leftarrow$ all possible histories

A Class of Recursive Rules

$\mathcal{H}$

$$\mathcal{H}^T = \left\{ (\pi^1, \ldots, \pi^K) \in \mathcal{H} : \sum_{k=1}^{K} \pi^k \in \Pi \right\}$$

Terminal histories where the cumulative allocation is a full (not partial) allocation

A Class of Recursive Rules

$\mathcal{H}$

$$\mathcal{H}^T = \left\{ (\pi^1, \ldots, \pi^K) \in \mathcal{H} : \sum_{k=1}^{K} \pi^k \in \Pi \right\}$$

$$\mathcal{H}^{NT} = \mathcal{H} \setminus \mathcal{H}^T \longleftarrow \text{non-terminal histories}$$

A Class of Recursive Rules

$$\mathcal{H}$$

$$\mathcal{H}^T = \left\{ (\pi^1, \dots, \pi^K) \in \mathcal{H} : \sum_{k=1}^{K} \pi^k \in \Pi \right\}$$

$$\mathcal{H}^{NT} = \mathcal{H} \setminus \mathcal{H}^T$$

$$\sigma : \mathcal{H}^{NT} \longrightarrow \mathcal{Y} \longleftarrow \text{sequencing rule}$$

A Class of Recursive Rules

That's a lot to keep track of

A Class of Recursive Rules

That's a lot to keep track of

Summary:

$f \in \mathcal{F}$ – solves a little bit of the allocation problem

A Class of Recursive Rules

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$f \in \mathcal{F}$ – solves a little bit of the allocation problem

$\eta \in \mathcal{H}$ – keeps track of the small steps

A Class of Recursive Rules

That's a lot to keep track of

Summary:

$f \in \mathcal{F}$ – solves a little bit of the allocation problem

$\eta \in \mathcal{H}$ – keeps track of the small steps

$\sigma : \mathcal{H} \to \mathcal{F}$ – says how the next bit should be solved

A Class of Recursive Rules

Let's put those parts together

A Class of Recursive Rules

Let's put those parts together

$$\Phi(\eta, u, s) =$$

given the steps so far

and utilities

and what's left to allocate

A Class of Recursive Rules

Let's put those parts together

$$\Phi(\eta, u, s) = \begin{cases} \vec{0} & \text{if } \eta \in \mathcal{H}^T \\ & \end{cases}$$

There's nothing left to do

A Class of Recursive Rules

Let's put those parts together

$$\Phi(\eta, u, s) = \begin{cases} \vec{0} & \text{if } \eta \in \mathcal{H}^{T} \\ & \text{if } \eta \in \mathcal{H}^{NT} \end{cases}$$

$\sigma(\eta)(s, u)$

This is the partial rule we're supposed to use

A Class of Recursive Rules

Let's put those parts together

$$\Phi(\eta, x, s) = \begin{cases} \vec{0} & \text{if } \eta \in \mathcal{H}^{T} \\ & \text{if } \eta \in \mathcal{H}^{NT} \\ \pi = \sigma(\eta)(s, x) & \end{cases}$$

A Class of Recursive Rules

Let's put those parts together

$$\Phi(\eta, u, s) = \begin{cases} \vec{o} & \text{if } \eta \in H^{T} \\ \eta \mathbin{+\!\!+} \pi & \text{if } \eta \in H^{NT} \\ \pi = \sigma(\eta)(s, u) & \end{cases}$$

Collect this step in the history

A Class of Recursive Rules

Let's put those parts together

$$\Phi(\eta, \nu, s) = \begin{cases} \vec{o} & \text{if } \eta \in \mathcal{H}^{T} \\ \eta \mathbin{++} \pi \quad r(s, \pi) & \text{if } \eta \in \mathcal{H}^{NT} \\ \pi = \sigma(\eta)(s, \nu) & \end{cases}$$

That's what is left over after this step

A Class of Recursive Rules

Let's put those parts together

$$\Phi(\eta, u, s) = \begin{cases} \vec{0} & \text{if } \eta \in \mathcal{H}^{T} \\ \eta +\!\!+\, \pi \quad r(s, \pi) & \text{if } \eta \in \mathcal{H}^{NT} \\ \pi = \sigma(\eta)(s, u) & \end{cases}$$

A Class of Recursive Rules

Let's put those parts together

$$\Phi(\eta, u, s) = \begin{cases} \vec{0} & \text{if } \eta \in \mathcal{H}^{T} \\ \Phi(\eta +\!\!+ \pi, u, r(s,\pi)) & \text{if } \eta \in \mathcal{H}^{NT} \\ \pi = \sigma(\eta)(s,u) & \end{cases}$$

recursively solve the remaining problem

A Class of Recursive Rules

Let's put those parts together

$$\Phi(\eta, u, s) = \begin{cases} \vec{0} & \text{if } \eta \in \mathcal{H}^{T} \\ \pi + \Phi(\eta{+}{+}\pi, u, r(s,\pi)) & \text{if } \eta \in \mathcal{H}^{NT} \end{cases}$$

where

$$\pi = \sigma(\eta)(s, u)$$

return the current step plus what the recursive call returns

A Class of Recursive Rules

Let's put those parts together

$$\Phi(\eta, u, s) = \begin{cases} \vec{0} & \text{if } \eta \in H^T \\ \pi + \Phi(\eta{+}{+}\pi, u, r(s,\pi)) & \text{if } \eta \in H^{NT} \\ \quad \text{where} & \\ \quad \pi = \sigma(\eta)(s,u) & \end{cases}$$

$$\phi(u) = \Phi(\langle\rangle, u, \vec{1})$$

Start with the empty history and full supply

A Class of Recursive Rules

Example: Serial Dictatorship (Svensson 1999)

A Class of Recursive Rules

Example: Serial Dictatorship (Svensson 1999)

Number the agents 1 to n

A Class of Recursive Rules

Example: Serial Dictatorship (Svensson 1999)

Number the agents 1 to n

$f^i \leftarrow$ partial allocation rule that gives i their best object with probability 1

A Class of Recursive Rules

Example: Serial Dictatorship (Svensson 1999)

Number the agents 1 to n

$f^i \leftarrow$ partial allocation rule that gives i their best object with probability 1

$$\sigma(\eta) = f^{\text{length}(\eta)}$$

A Class of Recursive Rules

Almost there.

A Class of Recursive Rules

Almost there.

Each of the f_i's is a monarchy

A Class of Recursive Rules

Almost there.

Each of the f^i's is a monarchy

Given i, j and $\alpha \in (0,1)$ a diarchy is

$f^{ij\alpha} \longleftarrow$ with α weight i pick first and then j and the opposite with weight $(1-\alpha)$

A Class of Recursive Rules

Almost there.

Each of the f^i's is a monarchy

Given i, j and $\alpha \in (0,1)$ a diarchy is

$f^{ij\alpha}$

$\mathcal{D} \leftarrow$ all diarchies

A Class of Recursive Rules

Almost there.

Each of the f^i's is a monarchy

Given i, j and $\alpha \in (0,1)$ a diarchy is

$f^{ij\alpha}$

$\mathcal{D} \supseteq \mathcal{M}$

includes monarchies $(i=j)$

A Class of Recursive Rules

Hierchy of monarchs and diarchs

$$\sigma(\eta) \in \mathcal{D}$$

A Class of Recursive Rules

Hierchy of monarchs and diarchs

$$\sigma(\eta) \in \mathcal{D}$$

Conditions:

- If $\sum_{\pi^k \in \eta} \pi^k$ is not integral then $\sigma(\eta) \in \mathcal{M}$
- Depends only on who gets something in η and whether they got integral or fractional allocations

A Characterization

ϕ is a HMD $\Rightarrow$ ϕ is efficient
strategy-proof
non-bossy
ordinal
neutral

A Characterization

Add an axiom to the list

ϕ is **bounded ly invariant** if $\forall u \in \mathcal{U}^N$, $\forall i \in N$, $\forall a \in A$
$\forall u_i' \in \mathcal{U}$

if $\{b : u_{ib} \geq u_{ia}\} = \{b : u'_{ib} \geq u'_{ia}\}$

A Characterization

Add an axiom to the list

ϕ is boundedly invariant if $\forall u \in \mathcal{U}^N$, $\forall i \in N$, $\forall a \in A$

$\forall u'_i \in \mathcal{U}$

if $\{b : u_{ib} \geq u_{ia}\} = \{b : u'_{ib} \geq u'_{ia}\} = B$

and

$\forall b \in B \quad u_{ib} = u'_{ib}$

then $\forall j$, $\phi_{ja}(u) = \phi_{ja}(u'_i, u_{-i})$

A Characterization

Add an axiom to the list

ϕ is bounded**ly invariant** if $\forall u \in \mathcal{U}^N$, $\forall i \in N$, $\forall a \in A$ $\forall u_i' \in \mathcal{U}$

$$\text{if } \left.\begin{array}{c} \{b : u_{ib} \geq u_{ia}\} = \{b : u'_{ib} \geq u'_{ia}\} = B \\ \text{and} \\ \forall b \in B \quad u_{ib} = u'_{ib} \end{array}\right\} \text{ then } \forall j,\ \phi_{ja}(u) = \phi_{ja}(u'_i, u_{-i})$$

Bogomolnaia & Heo (2012)

A Characterization

Theorem:

ϕ is a HMD $\iff$ ϕ is

- efficient
- strategy-proof
- non-bossy
- ordinal
- neutral
- boundedly invariant

A Characterization

Theorem:

ϕ is a HMD $\iff$ ϕ is efficient
strategy-proof
non-bossy
ordinal
neutral
boundedly invariant

An extension of Svensson's (1999) characterization of serial dictatorship in determinstic model.

Recap

- Care is needed in adapting ideas to new settings

Recap

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- We think our formulation of efficiency is the consistent one

Recap

- Care is needed in adapting ideas to new settings

- We think our formulation of efficiency is the consistent one

- With strategy-proofness there is a significant limitation on randomization when we impose efficiency

Important Open Questions

- Better understanding of how weak sd-efficiency really is

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- Characterization like ours, but with sd-efficiency

Important Open Questions

- Better understanding of how weak sd-efficiency really is
- Characterization like ours, but with sd-efficiency
- Ordinal rules that are "more fair" if we drop strategy-proofness?

Thanks!